Statics Aware Grid Shells: Additional Material

Submission ID 1191

1 Density and Anisotropy: Common Assumptions

It is common knowledge, both in structural mechanics and structural optimization, that a structure which minimizes the strain energy under a prescribed set of forces is an *optimal structure*. Therefore we take the strain energy U as the reference parameter and then we use it to validate the structural performance of our structures.
Our structures are grid-shells composed of cells. Minimizing the global strain energy requires to minimize the strain energy locally for each individual cell. As a consequence, we assume that *the cells are our structure*, and then we try to derive:

1. the relationship $U(L)$ between the strain energy U and the isotropic hexagonal cell edge length L. We call this relationship *"density"*;
2. the relationship $U(\frac{F_y}{F_x}, \frac{b}{a})$ between the strain energy U and the couple of independent variables $\frac{F_y}{F_x}$ and $\frac{b}{a}$. Respectively the ratio between vertical and horizontal forces loading the hexagonal grid (their sum being constant) and the ratio between the radii of the ellipse circumscribed to the (generally anisotropic) hexagonal cell. We call this relationship *"anisotropy"*.

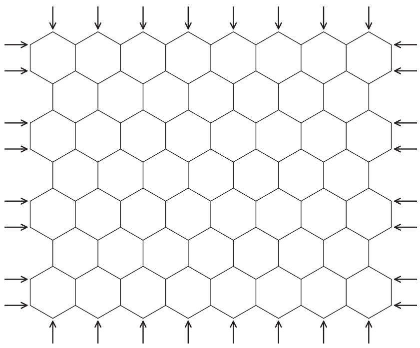
2 Density

In an isotropic hexagonal cell loaded as in Figure 1, each beam is loaded only at the nodes: therefore the analysis can be focused on a single beam only. For a planar beam of length L subject to in-plane loads only, the strain energy U is given by:

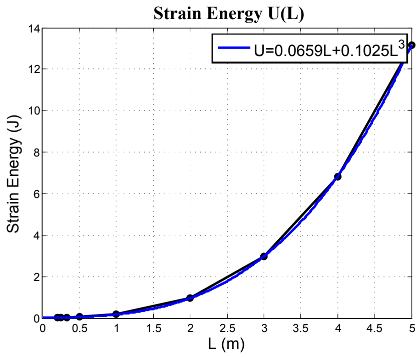
$$U = \frac{1}{2}\Big[\int_0^L N\epsilon + V\gamma + M\chi \; dz\Big] \tag{1}$$

where N is the axial force, ϵ the axial strain, V the shear, γ the shear strain, M the bending moment, χ the curvature and z the curvilinear abscissa of the beam.
Equation (1) can also be expressed in terms of the internal forces only:

$$U = \frac{1}{2}\Big[\int_0^L \frac{N^2}{EA} + \frac{V^2}{GA_v} + \frac{M^2}{EI} \; dz\Big] \tag{2}$$



(a) Isotropic Hexagonal Grid subjected to generic state of stress.



(b) Plot of the deformation energy as a function of the length L of the beams.

Figura 1: Hexagonal Grid, computation of Density Parameter.

where EA, GA_v, EI are respectively the axial stiffness, the shear stiffness and the bending stiffness of the beam, whereas A, A_v, I are respectively the area, shear area and moment of inertia of the cross section of the beam.
Since the beam is loaded only at the nodes by a generic force F, the internal forces can be expressed as:

$$\begin{aligned} N &= \alpha F \\ V &= \beta F \\ M &= (\gamma L - \delta z) \end{aligned} \tag{3}$$

so that the strain energy becomes:

$$\begin{aligned} U &= \frac{1}{2}\Big[\int_0^L \frac{(\alpha F)^2}{EA} + \frac{(\beta F)^2}{GA_v} + \frac{(\gamma L - \delta z)^2}{EI}\; dz\Big] \\ &= \frac{F^2}{2} L\Big[\frac{\alpha^2}{EA} + \frac{\beta^2}{GA_v} + \frac{(\gamma^2 + \frac{\delta^2}{3} - \gamma\delta)}{EI} L^2\Big] \\ &= \frac{F^2}{2}\big(\epsilon L + \zeta L^3\big) \end{aligned} \tag{4}$$

In most of the structural applications it happens that $\zeta > \epsilon$ and $\zeta L^3 >> \epsilon L$, it follows that the linear term can be neglected. Finally, the strain energy U proper of a beam which is loaded only at the nodes, takes the form:

$$U \propto F^2 L^3 \tag{5}$$

The same relation scales up to the whole grid of Figure 1a as well. Nevertheless, we have verified the result of equation (5) through a numerical experiment. We tiled a regular hexagonal pattern (see Figure 1a) and we assigned to all beams a uniform cross section (e.g. circular) and we applied a uniform force F acting on the whole boundary (e.g. the sum of all horizontal forces equals the sum of all vertical forces). Then we computed the global strain energy U for iteratively increasing edge lengths L: Figure 1b shows clearly that the function $U(L)$ is a cubic polynomial in L, whose coefficients are constant, as they rely only on beams' cross section (e.g. area A and moment of inertia I) and material (e.g.

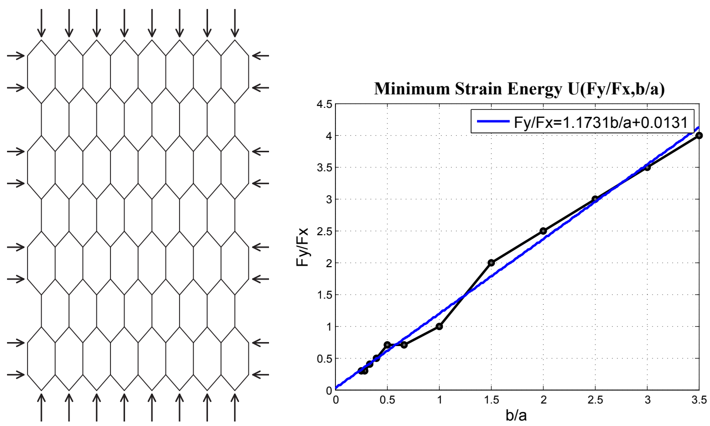
Young modulus E). This result confirms the validity of equation (5). It is also worth to notice that in Figure 1b the coefficient of the linear term ϵ is smaller than the coefficient of the cubic term ζ: this proves that our assumption of neglecting the linear term is valid.

3 Anisotropy

In an anisotropic hexagonal grid, loaded as in Figure 2a, the strain energy U depends both on the ratio $\frac{F_y}{F_x}$ between vertical and horizontal forces (their sum $F_y + F_x$ being constant) and on the ratio $\frac{b}{a}$ between the radii of the ellipse circumscribed to cell.

Unfortunately, outlining the relationship between the strain energy U and the couple of ratios $(\frac{F_y}{F_x})$ and $(\frac{b}{a})$ is not straightforward. Therefore we set up a numerical experiment where we compute the strain energy for several couples of ratios $(\frac{F_y}{F_x})$ and $(\frac{b}{a})$, paying attention to keep constant both the sum $F_y + F_x$ and the overall length of the grid L_{tot}. Figure 2b shows the experimental curve $C = min\big[U(\frac{F_y}{F_x}, \frac{b}{a})\big]$.

The relationship $min\big[U(\frac{F_y}{F_x}, \frac{b}{a})\big]$ is almost linear, and close to the trivial equation $\frac{F_y}{F_x} = \frac{b}{a}$. This means that the minimum of the global strain energy U is met if respectively the ratios between vertical and horizontal forces and between vertical and horizontal diameters of the ellipse circumscribed to an anisotropic cell, are the same.



(a) Anisotropic Hexagonal Grid subjected to generic state of stress.

(b) Plot of the anisotropy ratio $\frac{b}{a}$ of a stretched grid against the anisotropy ratio $\frac{F_y}{F_x}$ of the force field, yielding the minimal strain energies.

Figura 2: Hexagonal Grid, computation of Anisotropy Parameter.