From Point Clouds to tessellated surfaces

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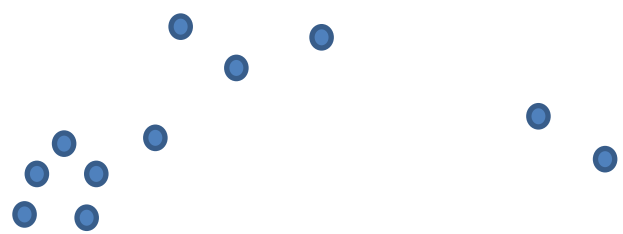
Problem Statement

Given a Point cloud $P = \{p_0, \dots, p_n\}, p_i \in \mathbb{R}^3$, find the mesh M that it *represents*



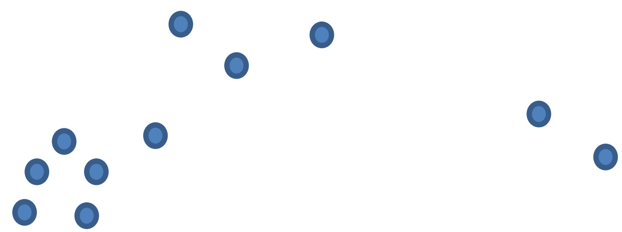
- Q1: It is a very ill posed problem, what does *represents* means?
- Q2: why do we care about this problem?

Motivations

- A1: Ideally, we want to find the surface which sampling produced the input problem
- A2: Every device or methods produces a discrete puntual sampling of the surface
  - **Laser scanning**
  - **Image based techniques**
  - **Computerized Axial Tomography / simulation data**

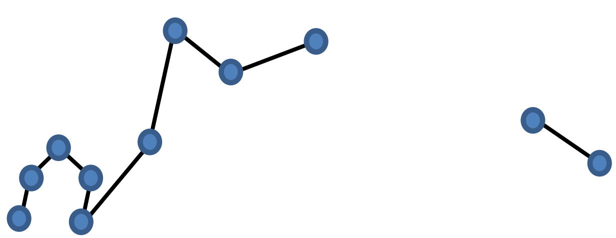
... So that is what we are dealing with

Explicit and Implicit Methods

$P =$ 

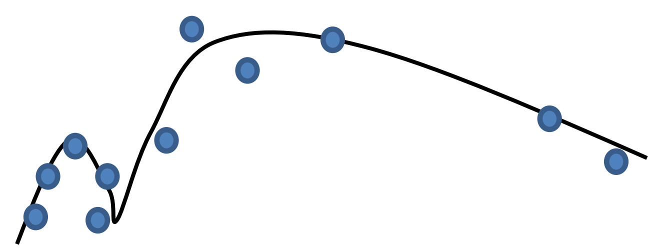
Explicit methods

Build a tessellation over the point cloud. The points map to vertices of the mesh



Implicit Methods

1. Define the surface implicitly, as the zeroes of a function $f_P: \mathbb{R}^3 \rightarrow \mathbb{R}^3$
2. Tessellate $\{f_P(x) = 0\}$



Explicit and Implicit Methods

Explicit methods

Build a triangulation over the point cloud. The points map to vertices of the mesh

- less robust to noise
- require a dense and even sampling
- Generally easier to implement

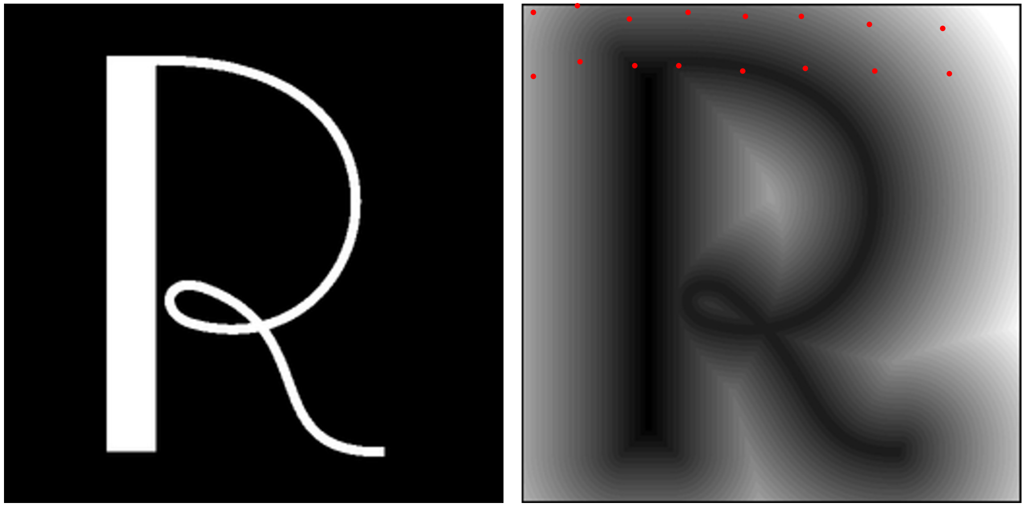
Implicit Methods

1. Define the surface implicitly, as the zeroes of a function $f_P\colon \mathbb{R}^3 \to \mathbb{R}^3$
2. Tessellate $\{f_P(x) = 0\}$

- more robust to noise
- more resilient to noise and uneven sampling

Volumetric methods

- **define a distance field** from the surface



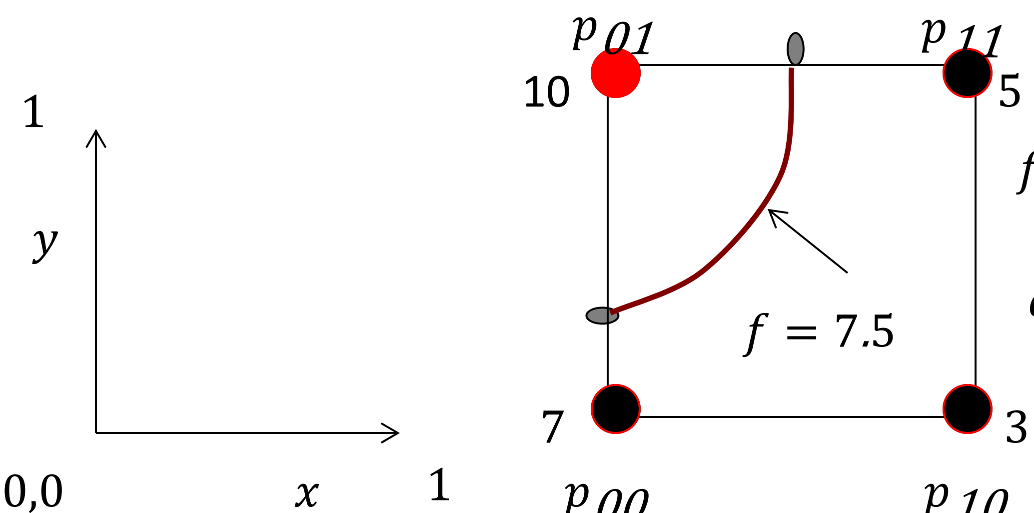
- return the **isosurface** for 0

Marching Cubes:isosurfaces from volume data [Lorensen87] :

Input:
- a regular 3D grid where each node is associated with a scalar value ***f*** (i.e. a scalar field)
- a scalar value α

Output: a surface with scalar value α and non null gradient (the isosurface)

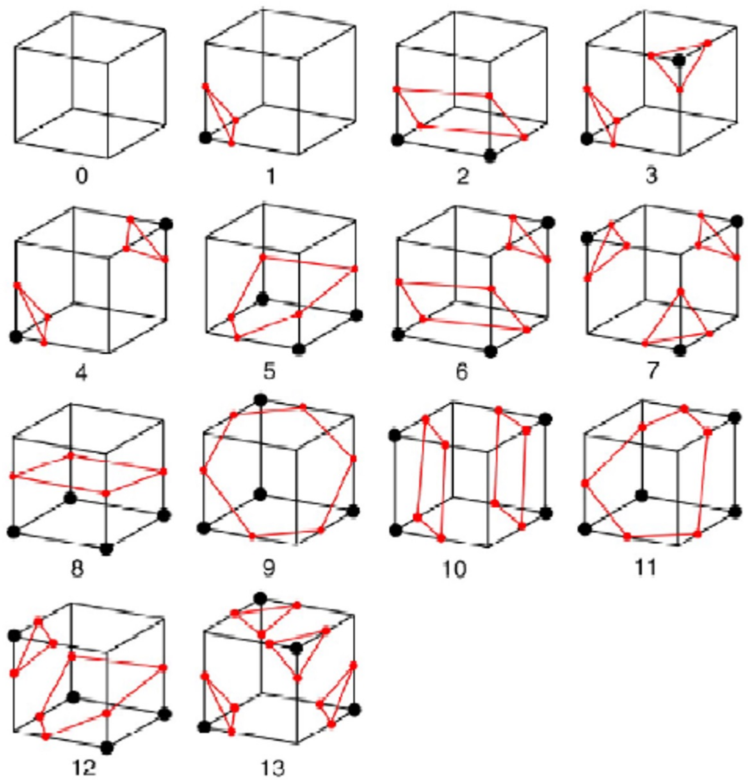
The value at p is obtained by trilinear interpolation of the values of the vertices of the grid cell contained in



$$f(x,y) = \sum_{i,j=0}^{i,j=1} f_{ij} \cdot (1-x)^{1-i} x^i (1-y)^{1-j} y^j$$

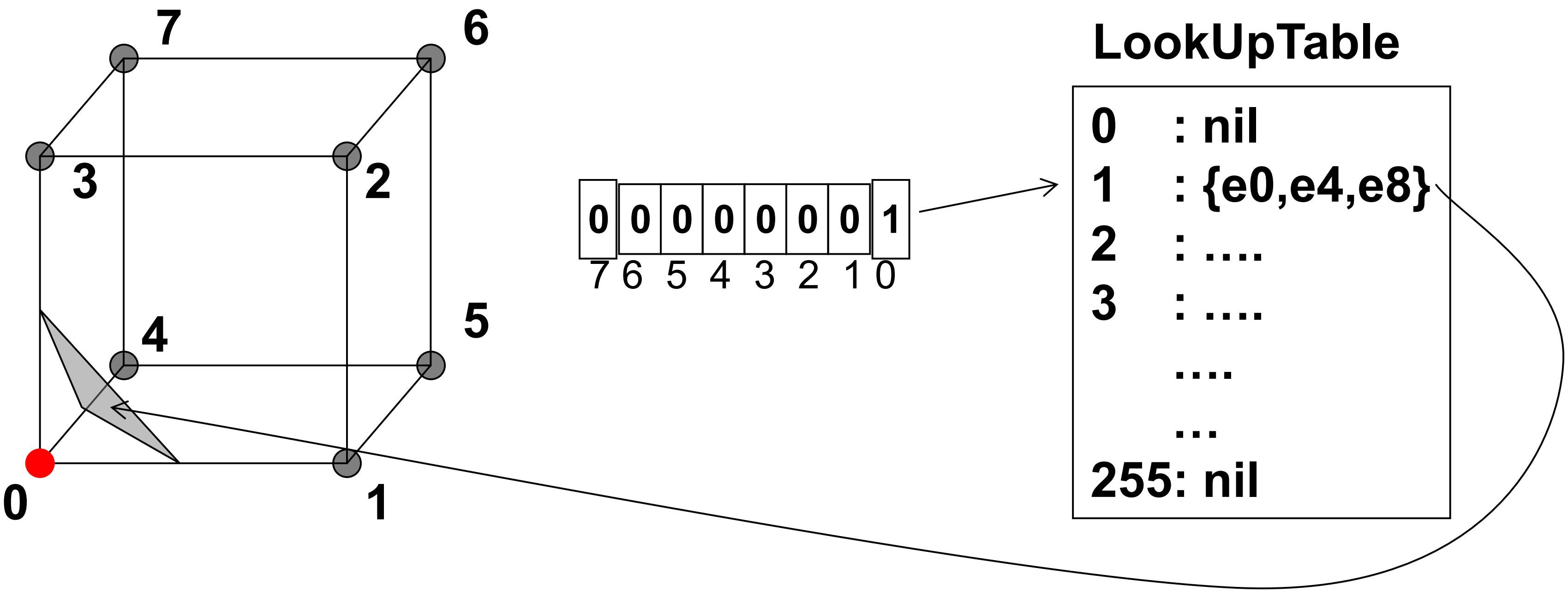
$\alpha = 7.5$

Marching Cube: configurations



- All configurations: 2^8=256, but only 14 considering rotations, mirroring and complement

Marching Cube: LookUp Table



For each combination of field value respect to the threshold, store the corresponding triangolation.

Marching Cubes: pros/issues

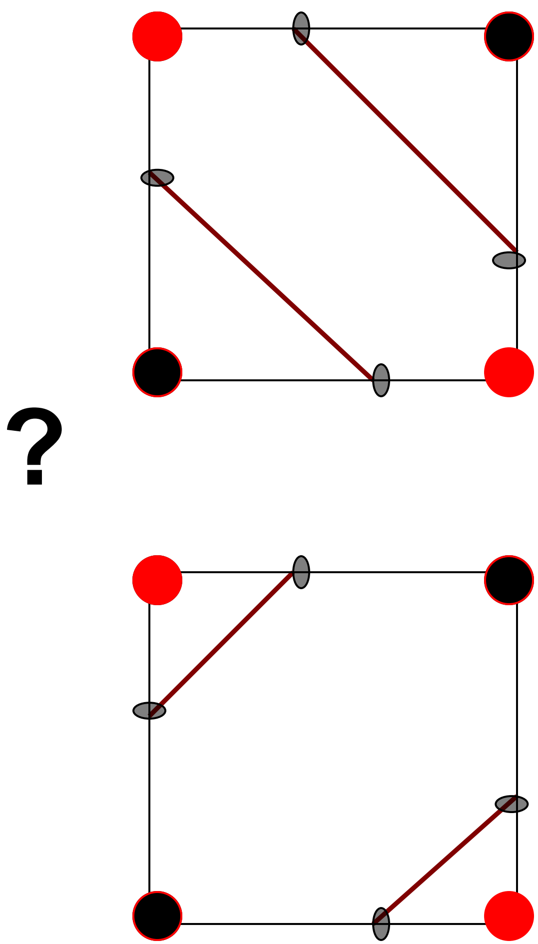
- Pros:
  - Quite easy to implement
  - Fast and not memory consuming
  - Very robust

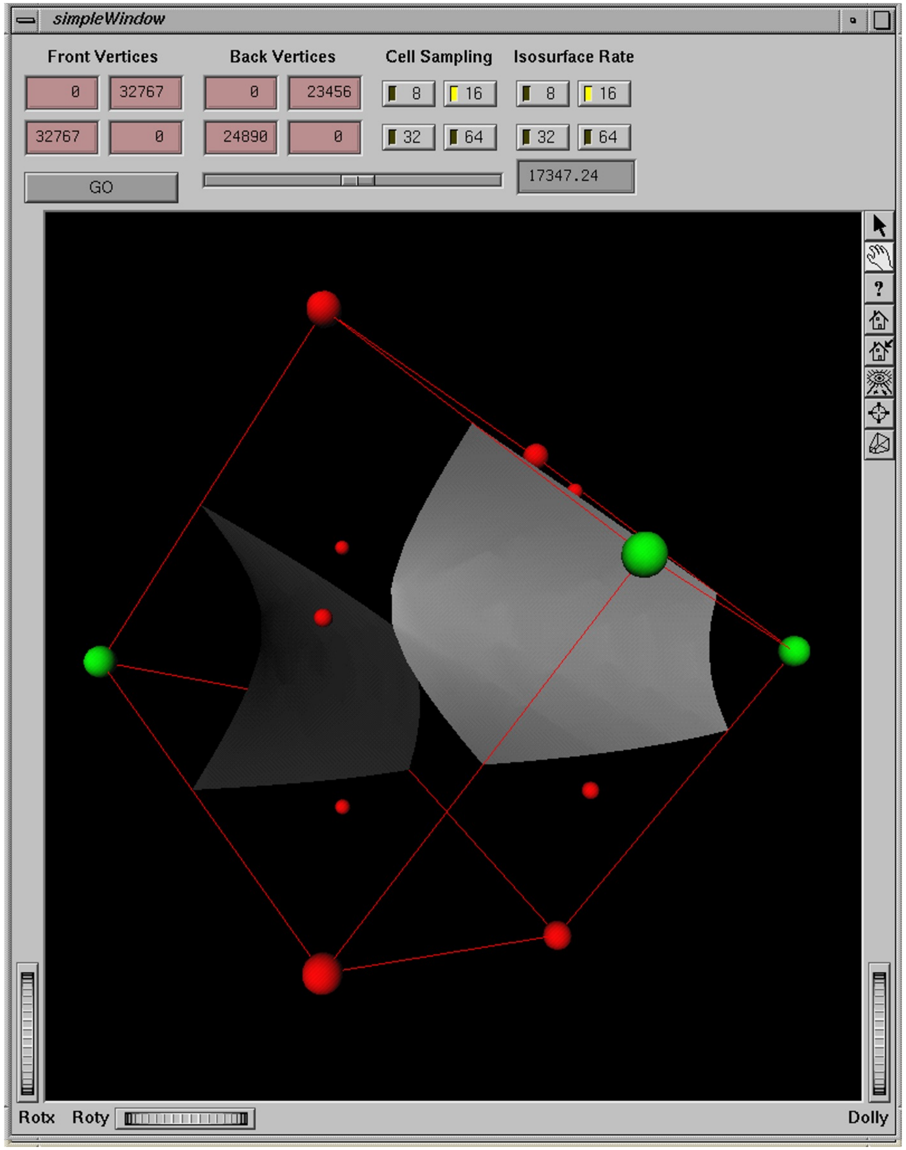
..then why from ‘87 zillions papers where published ?

Issues:

- **Consistency**. Guarantee a C0 and manifold result: ambiguous cases
- **Correctness**: return a good approximation of the “real” surface
- **Mesh complexity**: the number of triangles does not depend on the shape of the isosurface
- **Mesh quality**: arbitrarily ugly triangles

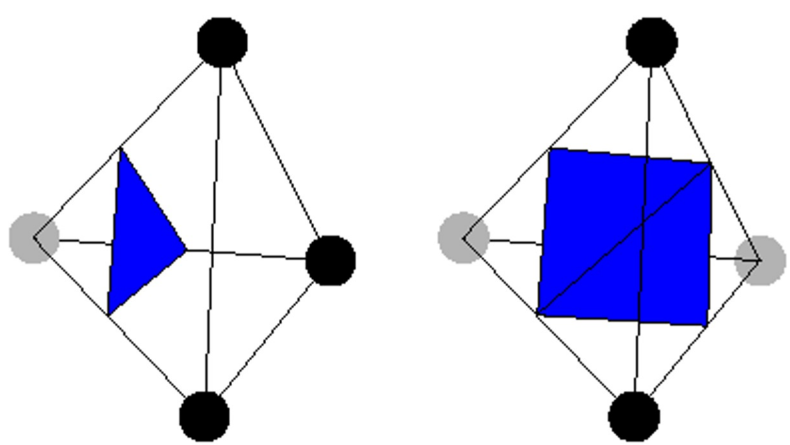
Marching Cubes: ambiguous cases





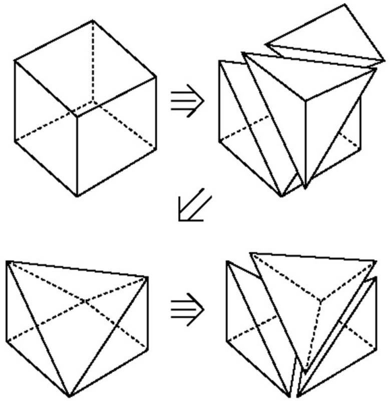
Marching Tetrahedra

- Tetrahedral cells (instead of cubical)
- Only 3 configurations (2^4 == 16 permutation of grid values reduce to 3 cases)
- No ambiguities (linear field on a value is planar)
- Boundary cases are easy to be managed too
  - Cases where you get a vertex with exact threshold value
- but it may be “less” correct
  - If you start from a cubic grid the tetrahedral decomposition is a biasing choice

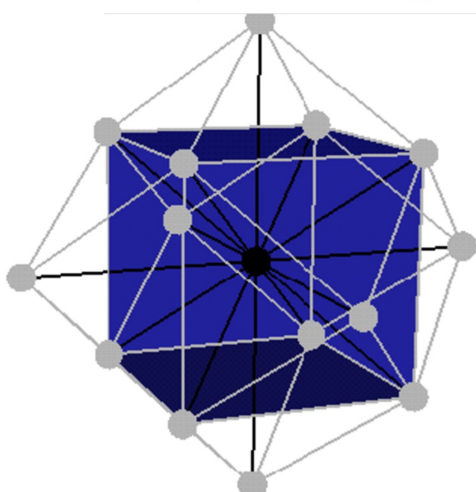


Marching Tetrahedra

- Original approach [Treece99]: cubic cells are partitioned in 5 (o 6) tetrahedra.
  - Subdivision determines topology



- Body centered cubic lattice: one more sample in the cubic cell
  - Unique subdivision
  - Equal tetrahedral
  - Better surface (better triangles)

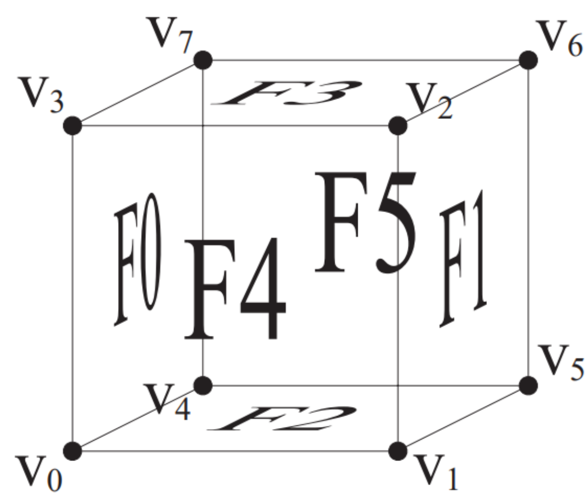


Resolving ambiguities

- The value of the scalar function inside each cell is interpolated by the (known) value of its 8 corners

$$T(x,y,z) = axyz + bxy + cyz + dxz + ex + fy + gz + h$$

$$\begin{aligned}
a &= v1 + v3 + v4 + v6 - v0 - v7 - v5 - v2 \\
b &= v0 + v2 - v1 - v3 \\
c &= v0 + v7 - v4 - v3 \\
d &= v0 + v5 - v1 - v4 \\
e &= v1 - v0 \\
f &= v3 - v0 \\
g &= v4
\end{aligned}$$



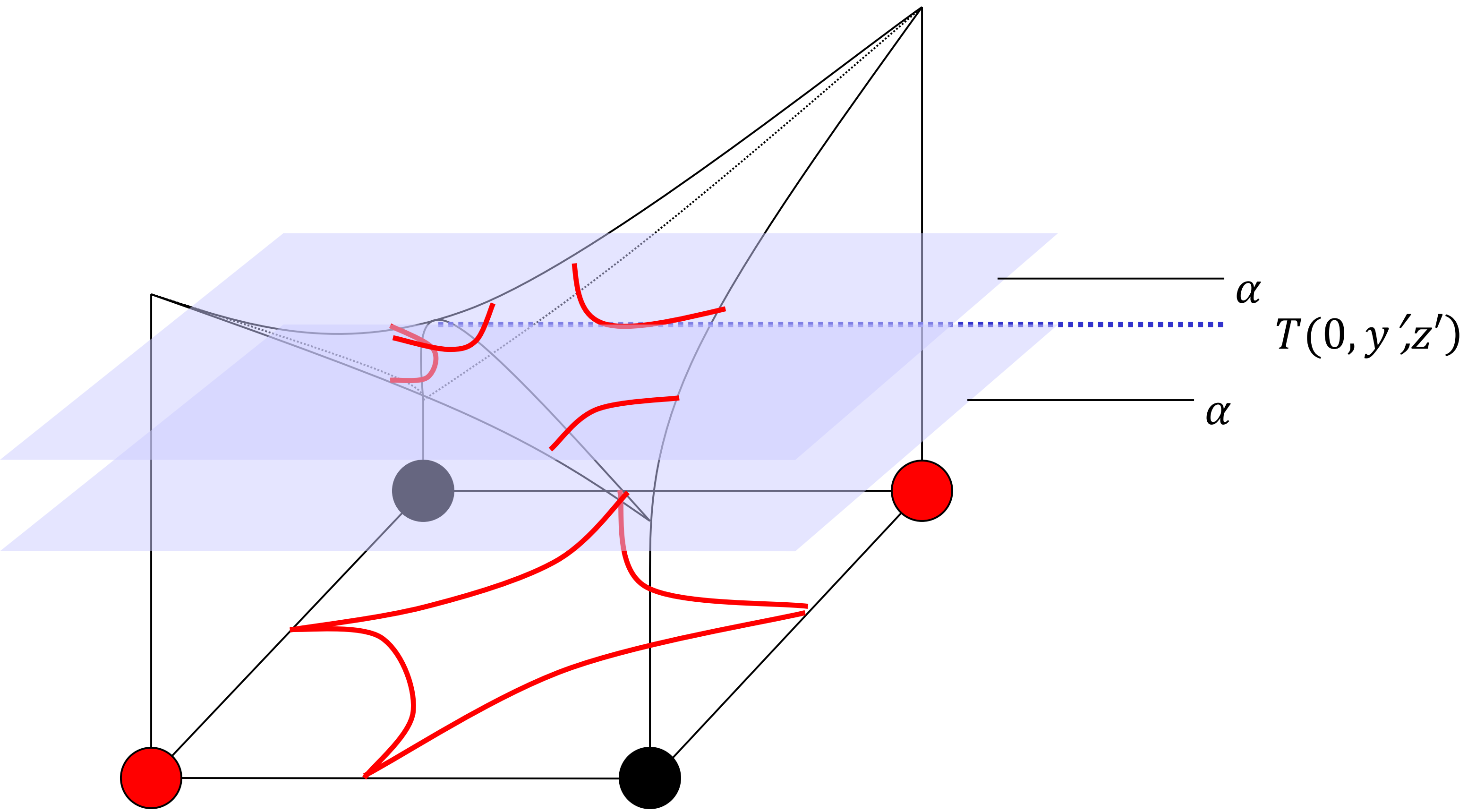
Saddle points

Field value on a cell's face

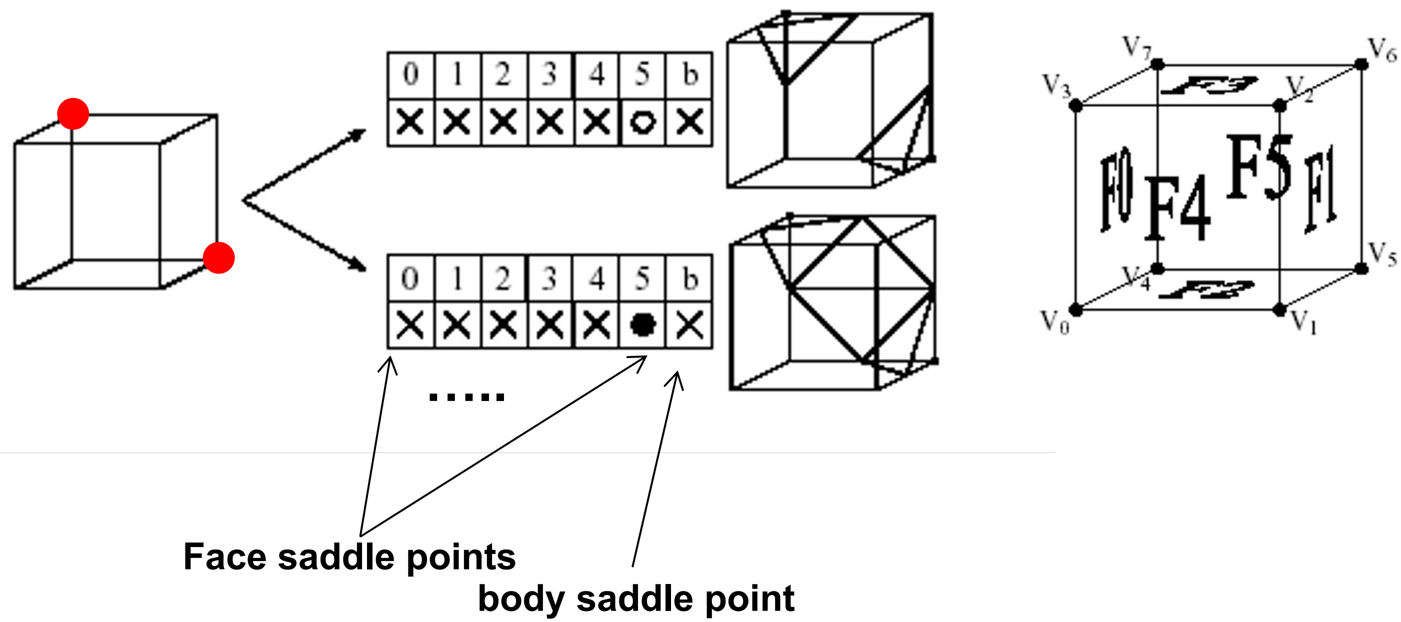
$$T(0, y, z) = c\,yz + f\,y + gz + h$$

$$\frac{\partial T(0, y', z')}{\partial y} = cz' + f = 0 \Rightarrow z' = -\frac{d}{c}$$

$$\frac{\partial T(0, y', z')}{\partial z} = cy' + g = 0 \Rightarrow y' = -\frac{g}{c}$$



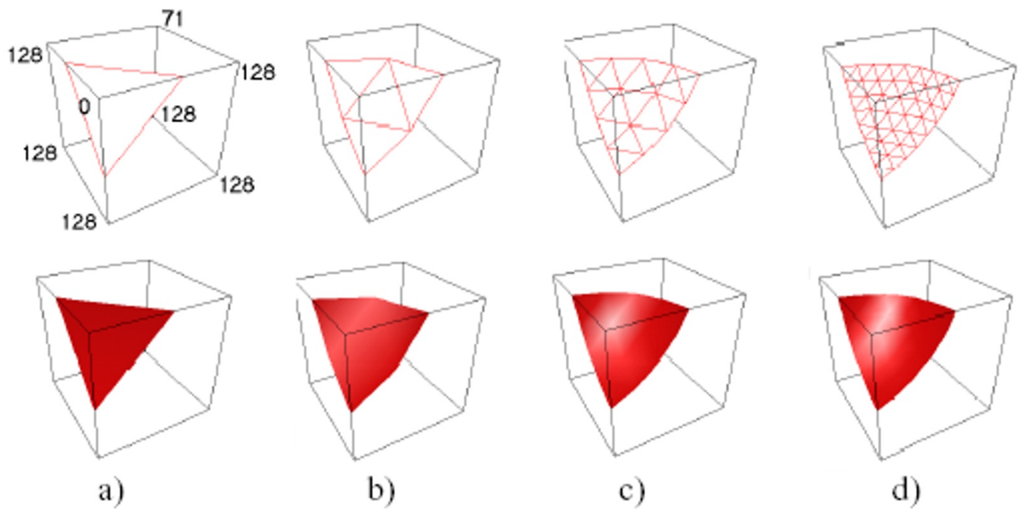
ELUT: Exhaustive LUT [Cignoni00]



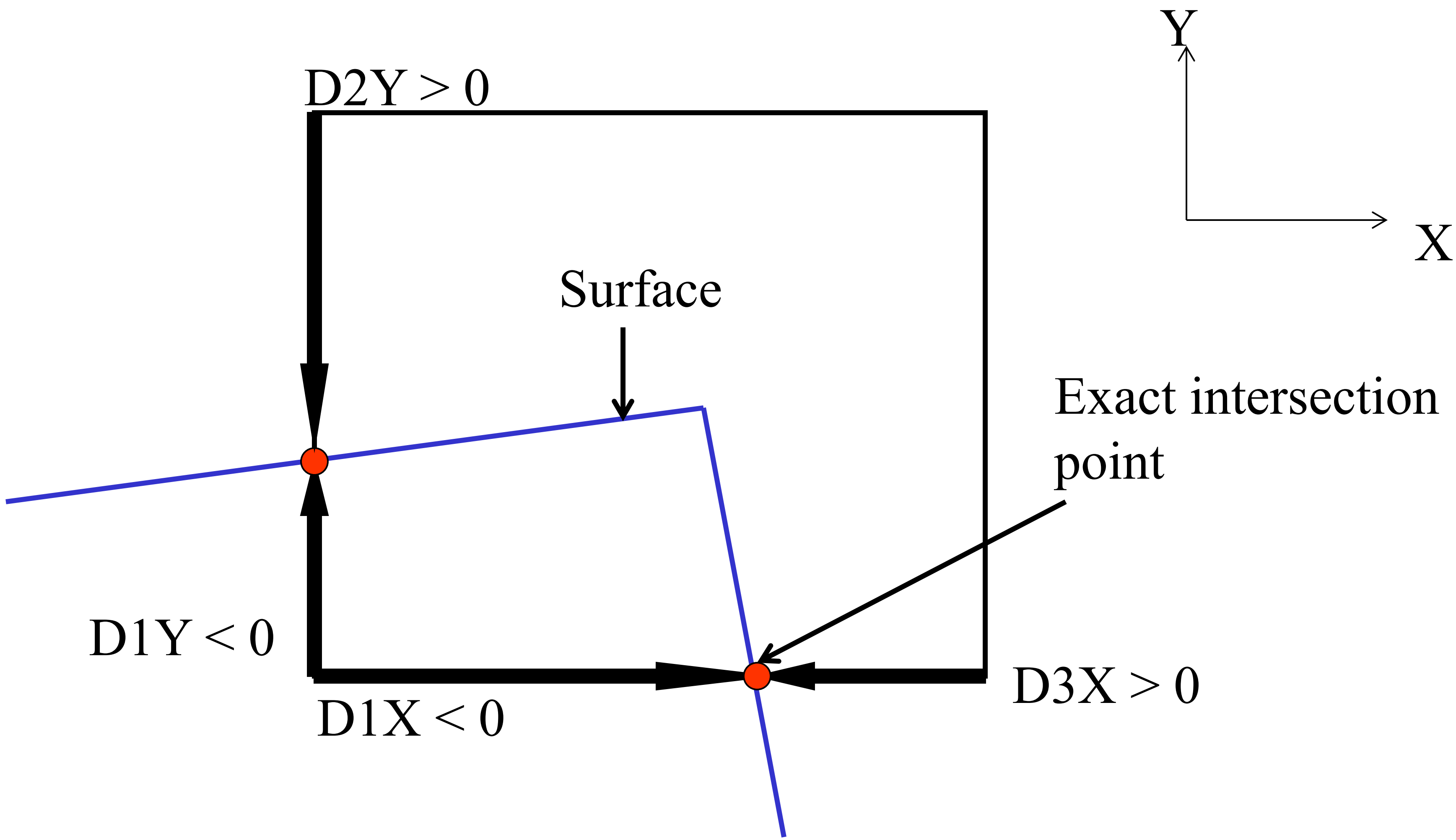
ELUT:
For each ambiguous configuration determines the coherent internal triangulation looking at the saddle points

Adaptive triangulation

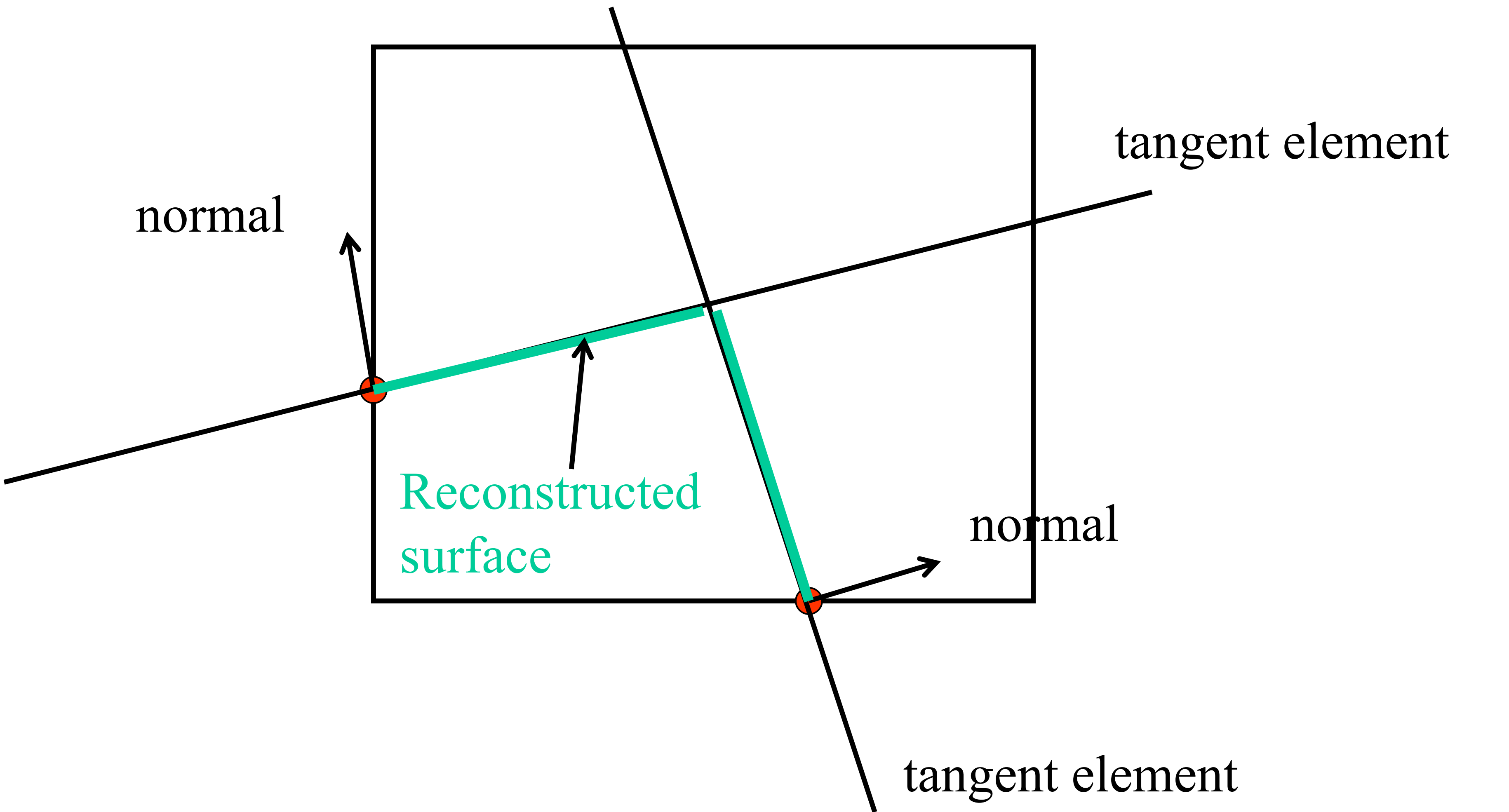
- Refine for better approximation (re-evaluate scalar field)



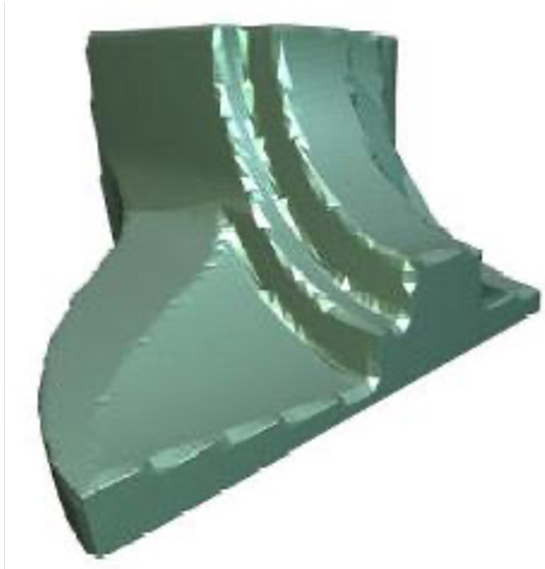
Extended MC [Kobbelt01]



MC


tangent element
normal
Reconstructed
surface
normal
tangent element

Extended MC

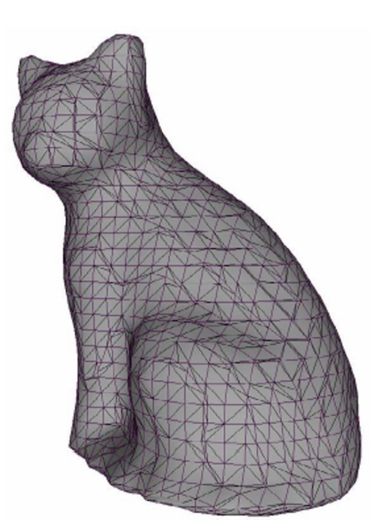


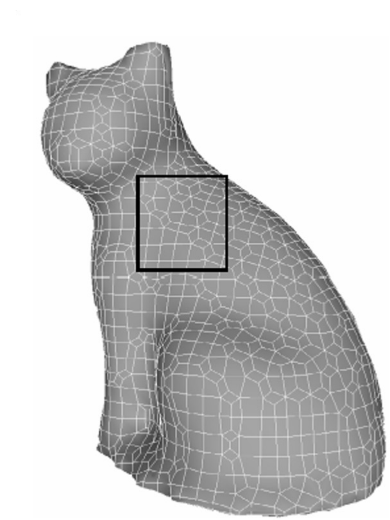
Marching Cubes

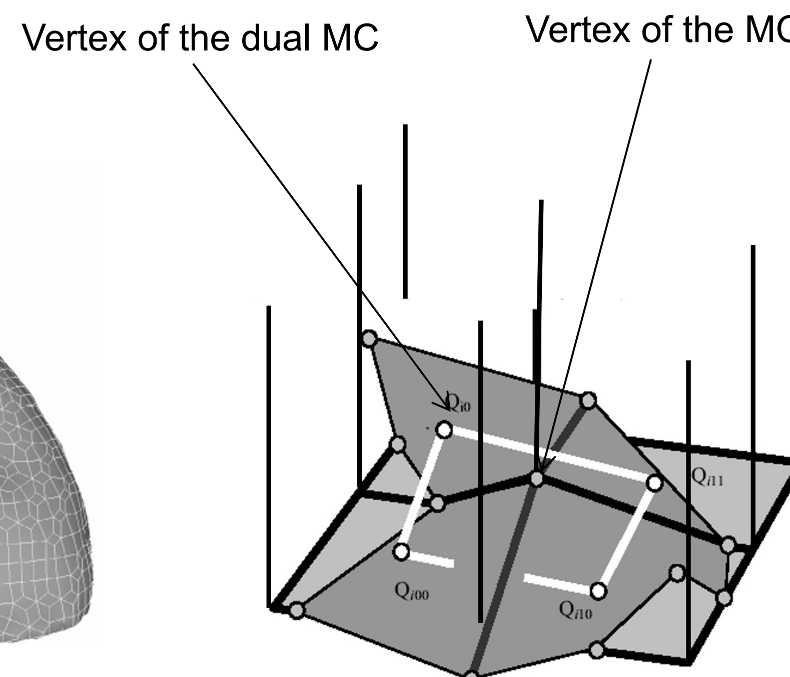
Extended Marching Cubes

Dual Marching Cubes [Nielson04]

- one vertex for each patch generated by MC
- One quad for each intersected edge (the 4 vertices associated to the patches of the cells sharing the edge)
- Tends to improve triangles quality

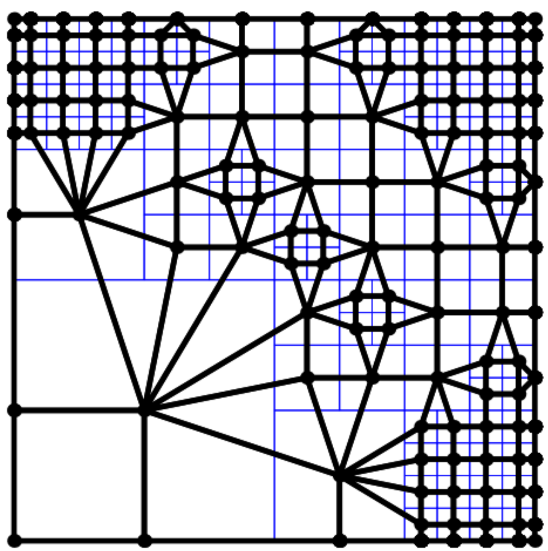


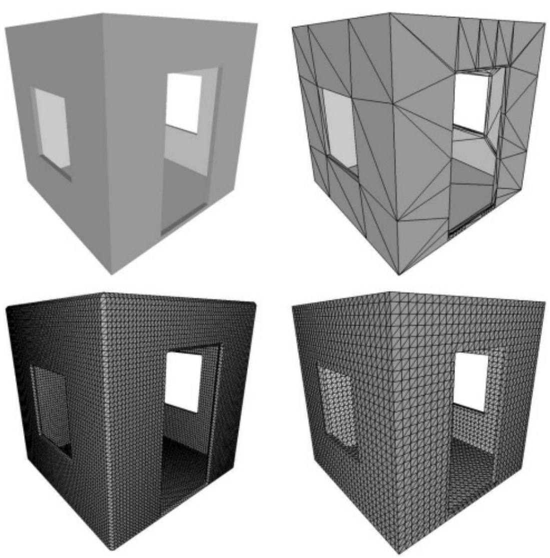




Dual Marching Cubes:Primal Contouring of Dual Grids [Shaeffer04]

- Partition the space with an Octree
- Build the dual grid
- Run MC on the dual grid (consider non hexahedral cells as HC with collapsed edges)





From point cloud to a scalar field…

Problem: given a set of points $\{x_0, \dots, x_n\}$, define

$$f(x) = \varphi(\{x_0, \dots, x_n\})$$

$$\mathrm{S} = \{\mathrm{x} \,|\mathrm{f(x)} = \alpha\}$$

so that S interpolates/approximates the point cloud

Normals are often either assumed or computed from the point cloud

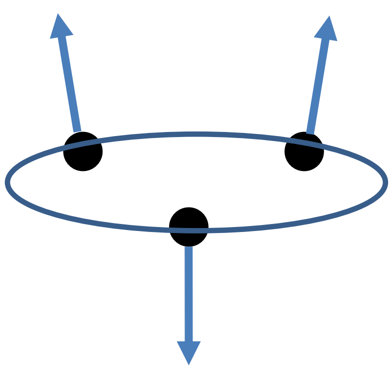
Point Cloud Normals (1/2)

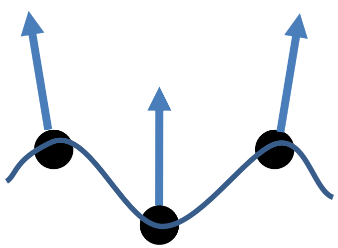
- Normals are important to define the surface

- Most of methods for building a surface from point cloud compute the normal on the points

Point Cloud Normals (1/2)

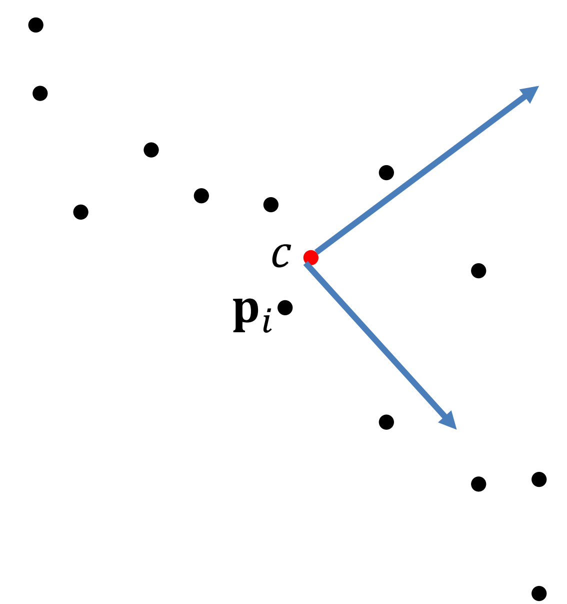
- Normals are important to define the surface





- Most of methods for building a surface from point cloud compute the normal on the points

Point Cloud Normals (2/2)



- Use Principal Component Analisys (PCA)

$$\mathbf{q}_i = \mathbf{p}_i - c$$

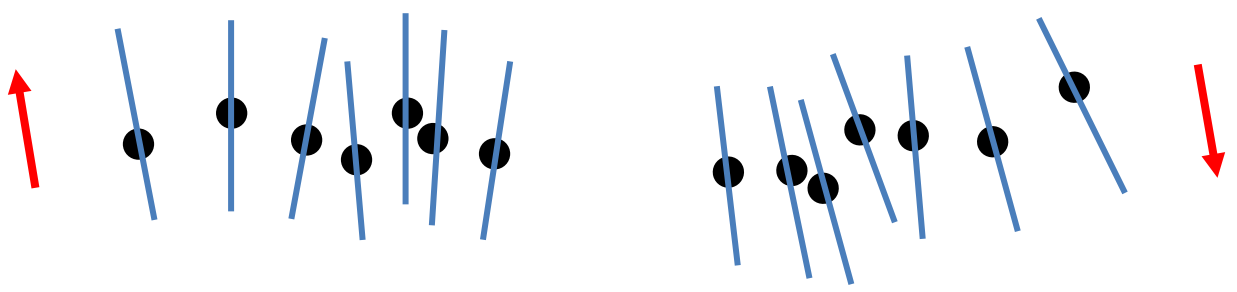
$$\mathbf{C}_{ov} = \sum_i \mathbf{q}_i \ \mathbf{q}_i^T$$

$$\mathbf{C}_{ov} = \begin{bmatrix} \sum_i q_{i_x}^2 & \sum_i q_{i_x} q_{i_y} & \sum_i q_{i_x} q_{i_z} \\ \sum_i q_{i_y} q_{i_x} & \sum_i q_{i_y}^2 & \sum_i q_{i_y} q_{i_z} \\ \sum_i q_{i_z} q_{i_x} & \sum_i q_{i_z} q_{i_y} & \sum_i q_{i_z}^2 \end{bmatrix}$$

- $\mathbf{C}_{ov}$ is symmetric → real eigenvalues and orthogonal eigenvectors
- take the eigenvector corresponding to the smallest eigenvalue as normal direction
  - Check that the smallest eigenvalue is unique
  - Check that the other two are similar

Point Cloud Normals (3/2)

- PCA is good for finding for each point a *direction* e.g. a line
- You have to choose a consistent alignment for groups of points
- Usually, heuristics based on sign propagation
  - Many difficult cases can arise for unconnected sets



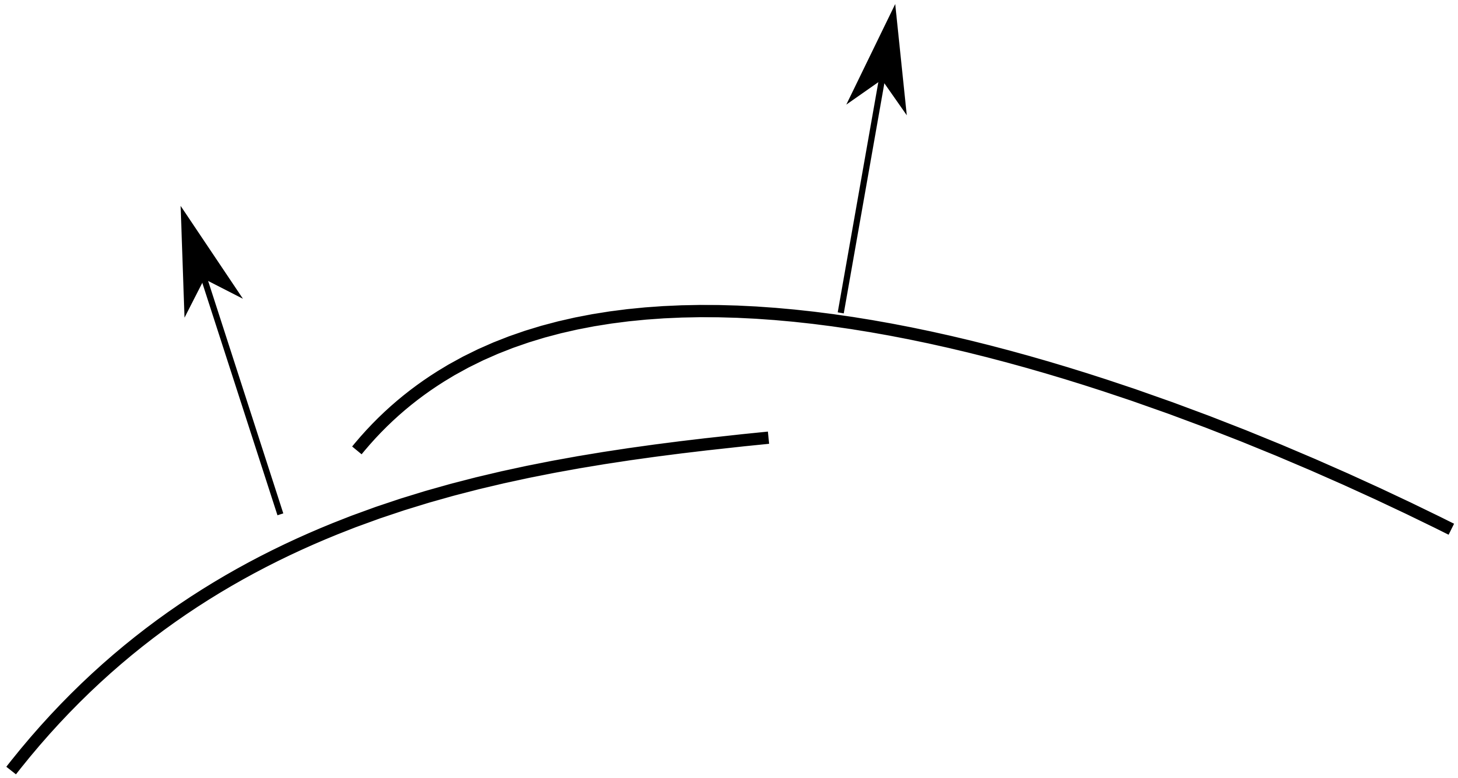
VCG Reconstruction/[Curless96]

- Suppose we do have aligned range maps
- We want to get a nice ISOSurface

1. Compute signed distance field from each range map
2. Average them
3. Extract the isosurface

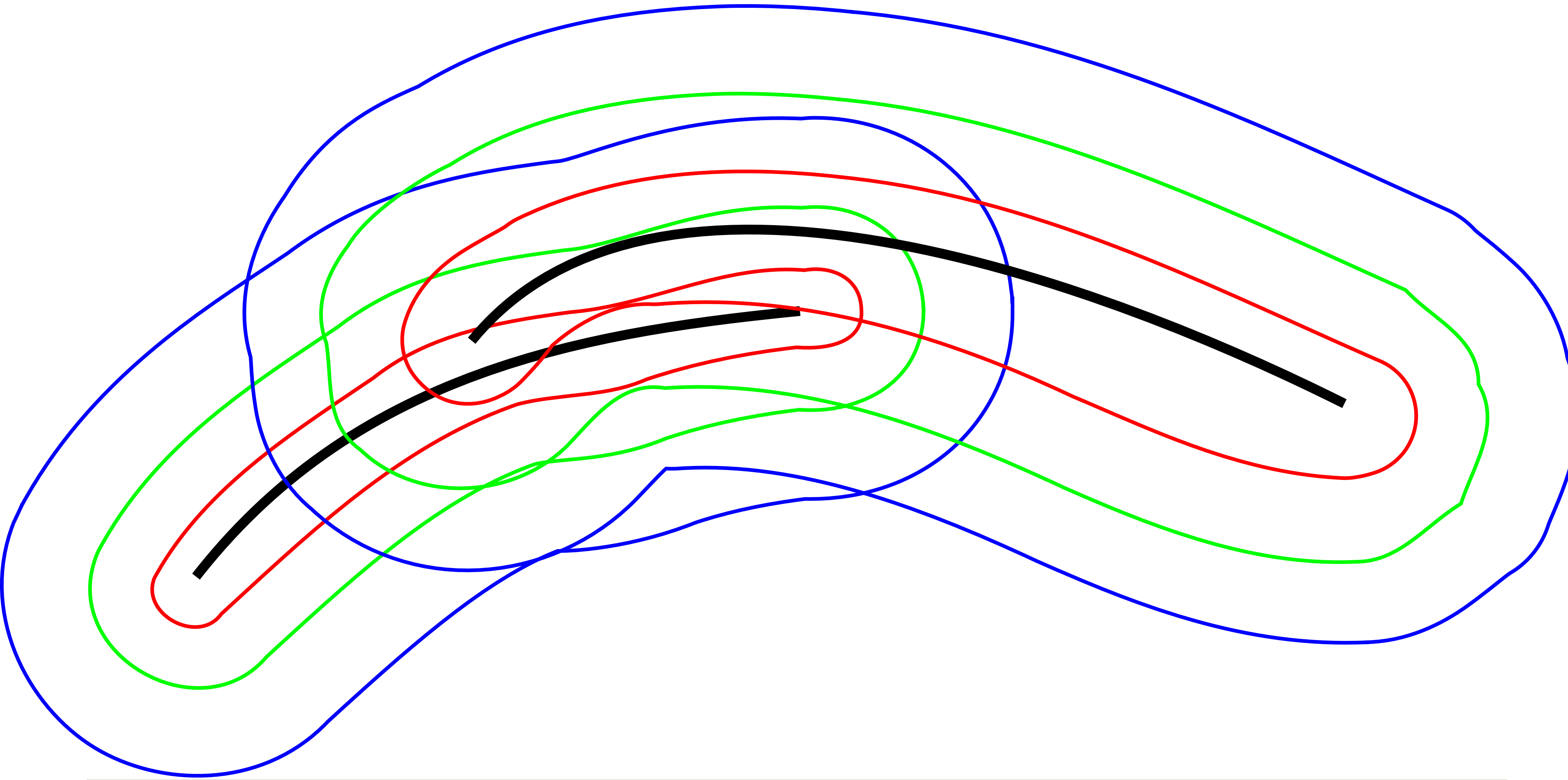
VCG Reconstruction/[Curless96]

- Surfaces with Normals



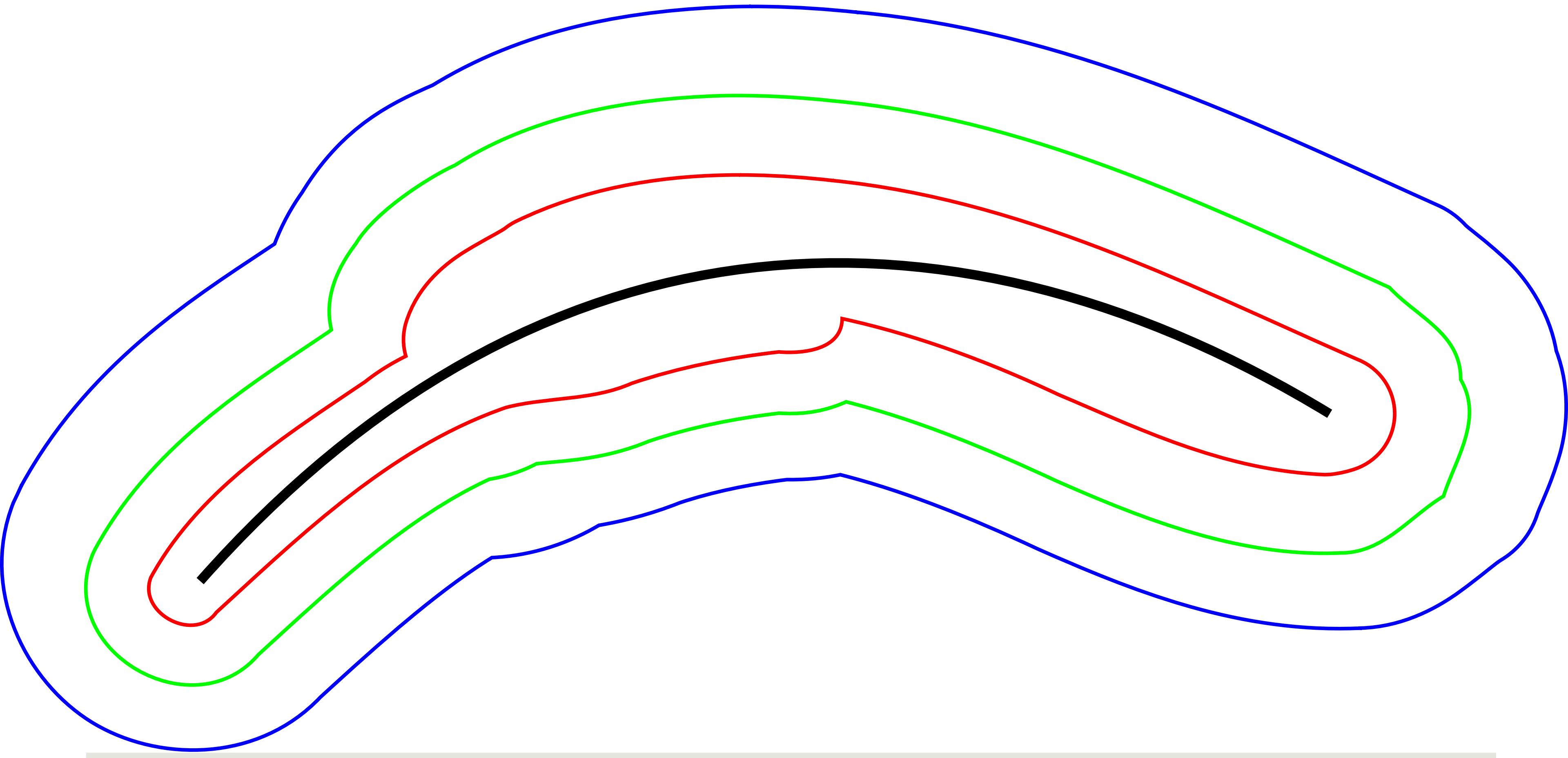
VCG Reconstruction/[Curless96]

- Compute Distance Fields (signed)



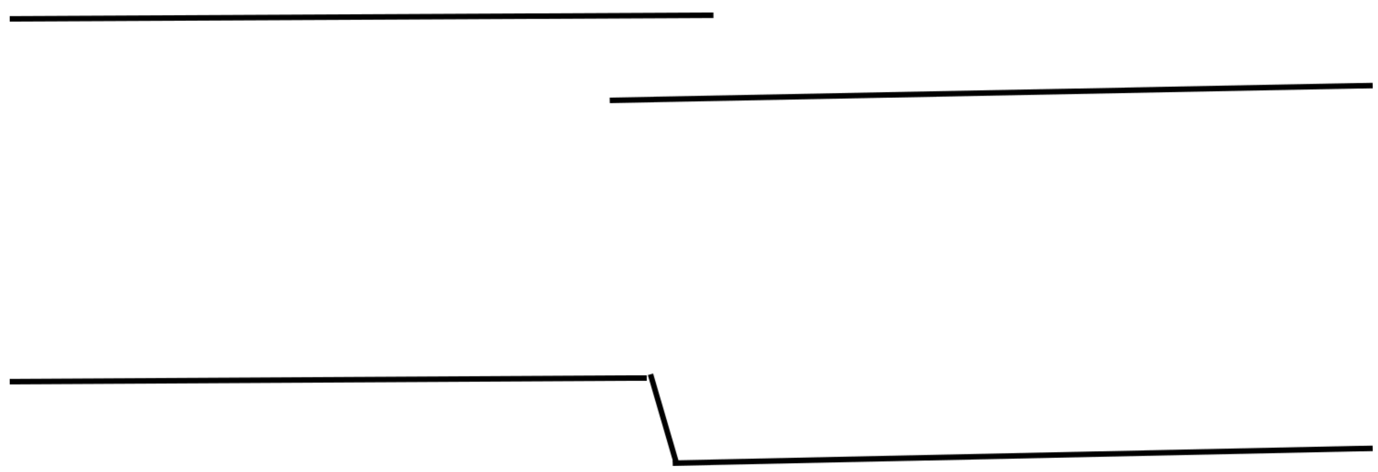
VCG Reconstruction/[Curless96]

- Average Distance Fields!



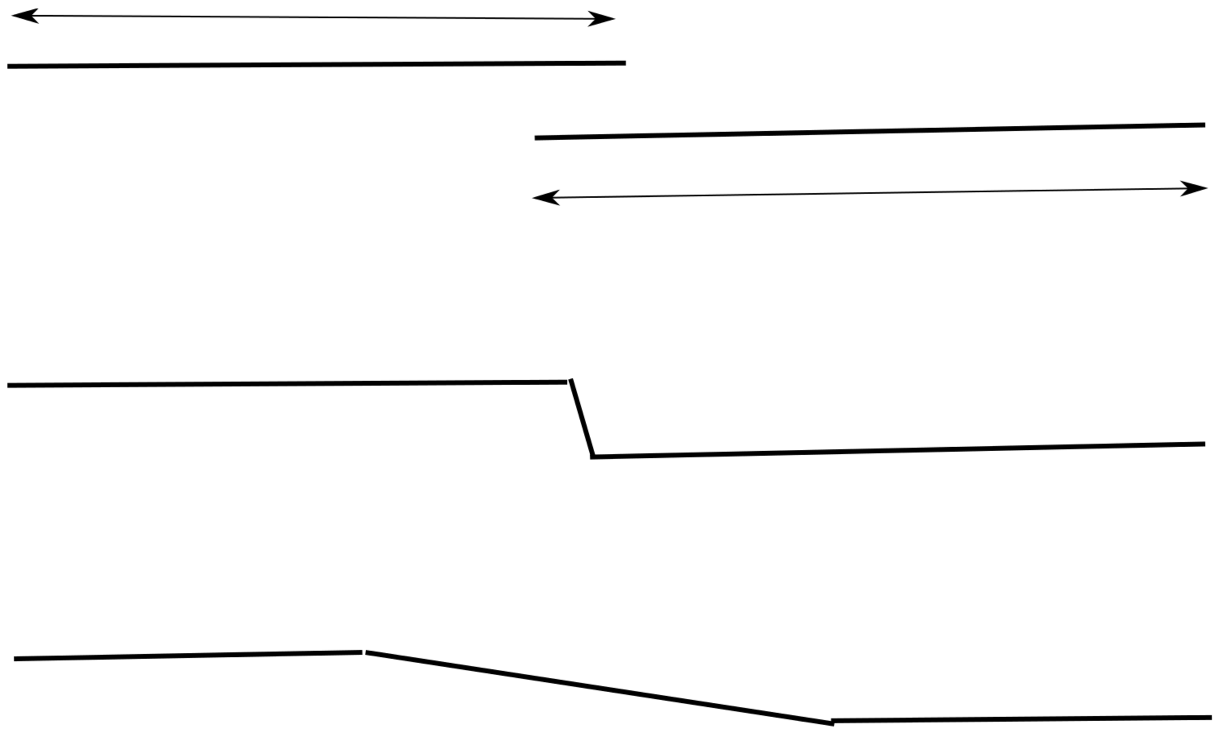
VCG Reconstruction: Issue

- This simple averaging can cause abrupt jumps



VCG Reconstruction (Use of geodesic)

- This simple averaging can cause abrupt jumps
- Solution: Weight the averaging by geodesic distance to border

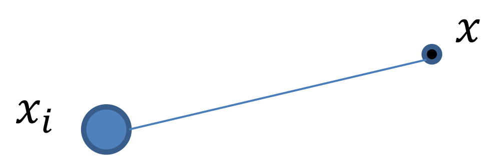


Metaballs [Blinnn92,Wyvill86]

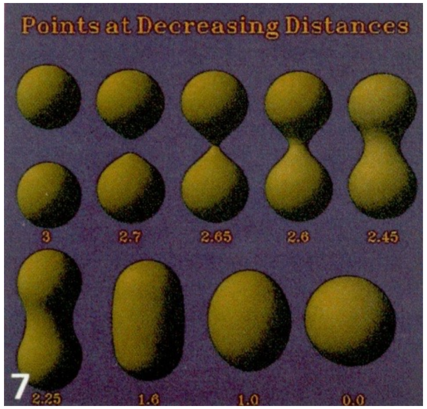
- f is the sum of function that have maximum in the points and decay with the distance

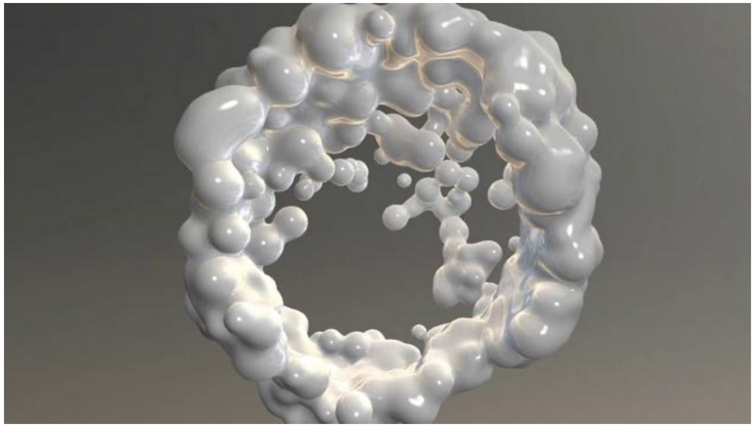
$$f(x_i) = 1 \quad f(R) = 0$$

$$f'(x_i) = 0 \quad f'(R) = 0$$



$$f(x) = \sum_i \left(2\frac{r^3}{R^3} - 3\frac{r^2}{R^2} + 1\right), r = \|x - x_i\|, R= \textbf{support radius}$$





Radial Basis Functions (RBF)

Solutions that follow the general scheme:

$$f(x) = \mathrm{p(x)} + \sum_i \omega_i \varphi(\|x - x_i\|)$$

$$f(x_i) = f_i$$

weights: $\omega_i \in \mathbb{R}$

RBF: $\varphi\colon \mathbb{R} \to \mathbb{R}$

p a polynome

Radial Basis Functions (RBF)[Carr01]

$$f(x) = \mathrm{p(x)} + \sum_i \omega_i \varphi(\|x - x_i\|),$$

$\omega_i \in \mathbb{R}$

$\varphi: \mathbb{R} \to \mathbb{R}$

p a polynome

$$\begin{bmatrix} A & \mathrm{P} \\ \mathrm{P}^T & 0 \end{bmatrix} \begin{bmatrix} \omega \\ c \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix}$$

$F = [f(x_1), \dots, f(x_N)]^T$
$A_{ij} = \varphi(\|x_j - x_i\|)$
p: basis for **all** polynomials of degree k
$P_{ij} = p_j(x_i)$

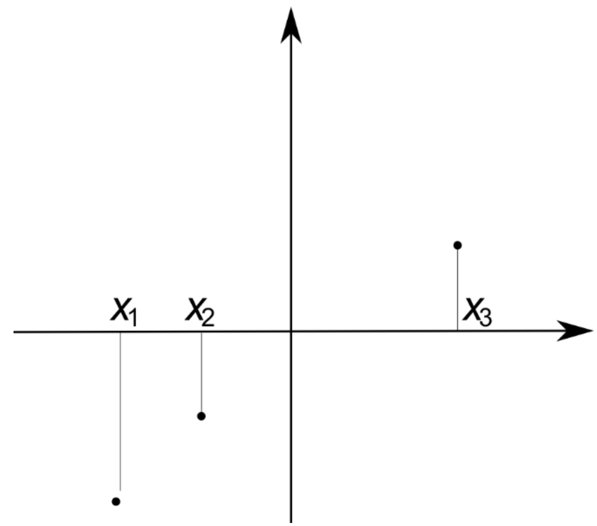
Examples of polynomial basis:

$p = \{1, x, y, z\}$ d=3, m=1

$p = \{1, x, y, x^2, xy, y^2\}$ d=2, m=2

$p = \{1, x, x^2, x^3\}$ d=1, m=3

Example



$x_1 = -2 \quad f_1 = -2$
$x_2 = -1 \quad f_2 = -1$
$x_3 = 2 \quad f_3 = 1$

Polynomial basis: {1,x}

$$P = \{1, x\}$$

$$\varphi(d) = d$$

$$\left[\begin{array}{ccc|cc}
\varphi(|x_1, x_1|) & \varphi(|x_1, x_2|) & \varphi(|x_1, x_3|) & p_1(x_1) & p_2(x_1) \\
\varphi(|x_2, x_1|) & \varphi(|x_2, x_2|) & \varphi(|x_2, x_3|) & p_1(x_2) & p_2(x_2) \\
\varphi(|x_3, x_1|) & \varphi(|x_3, x_2|) & \varphi(|x_3, x_3|) & p_1(x_3) & p_2(x_3) \\
\hline
p_1(x_1) & p_1(x_2) & p_1(x_3) & 0 & 0 \\
p_2(x_1) & p_2(x_2) & p_2(x_3) & 0 & 0
\end{array}\right]
\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ c_1 \\ c_2 \end{bmatrix}
=
\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ 0 \\ 0 \end{bmatrix}$$

Example

$$\left[\begin{array}{ccc|cc}
\varphi(|x_1,x_1|) & \varphi(|x_1,x_2|) & \varphi(|x_1,x_3|) & p_1(x_1) & p_2(x_1) \\
\varphi(|x_2,x_1|) & \varphi(|x_2,x_2|) & \varphi(|x_2,x_3|) & p_1(x_2) & p_2(x_2) \\
\varphi(|x_3,x_1|) & \varphi(|x_3,x_2|) & \varphi(|x_3,x_3|) & p_1(x_3) & p_2(x_3) \\
\hline
p_1(x_1) & p_1(x_2) & p_1(x_3) & 0 & 0 \\
p_2(x_1) & p_2(x_2) & p_2(x_3) & 0 & 0
\end{array}\right]
\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ c_1 \\ c_2 \end{bmatrix}
=
\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow$$

$$\begin{bmatrix} 0 & 1 & 4 & 1 & -2 \\ 1 & 0 & 3 & 1 & -1 \\ 4 & 3 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 & 0 \\ -2 & -1 & 2 & 0 & 0 \end{bmatrix}
\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ c_1 \\ c_2 \end{bmatrix}
=
\begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}
\quad \Rightarrow \quad
\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ c_1 \\ c_2 \end{bmatrix}
=
\begin{bmatrix} 0.125 \\ -0.166 \\ 0.0416 \\ -0.5 \\ 0.75 \end{bmatrix}$$

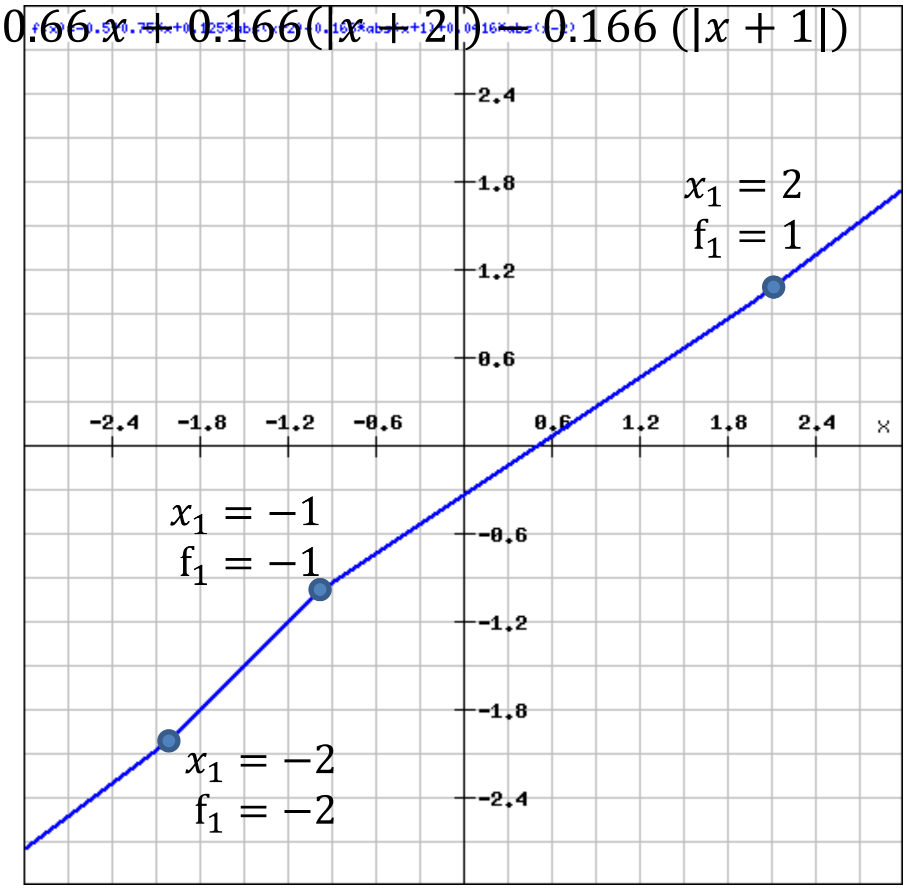
$$f(x) = -0.5 + 0.75\,x + 0.125\,|x+2| - 0.166|x+1| + 0.0416|x-2| =$$

$$= -0.334 + 0.66\,x + 0.166(|x+2|) - 0.166\,(|x+1|)$$

Example

$$f(x) = -0.5 + 0.75\,x + 0.125\,|x+2| - 0.166|x+1| + 0.0416|x-2|$$

$$= -0.334 + 0.66\,x + 0.166(|x+2|) - 0.166\,(|x+1|)$$



Radial Basis Functions (RBF)

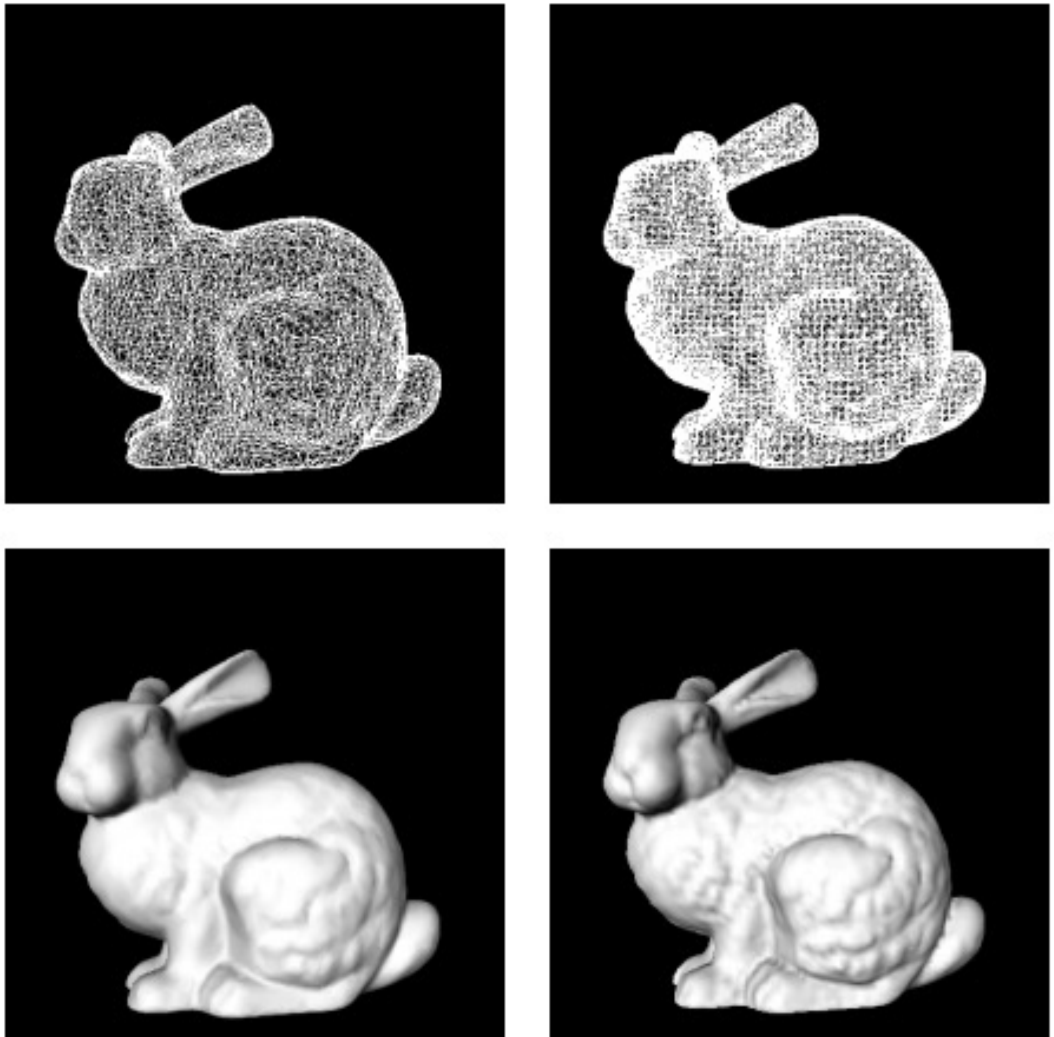
- Several possible choices for φ and *p:*
  - $\varphi(d) = d, linear\ polynomial$
  - $\varphi(d) = d^2, linear\ polynomial$
  - $\varphi(d) = d^3, linear/quadratic\ polynomial$
  - $\varphi(d) = d^2 \log(d), linear/quadratic\ polynomial$
  - ….
- Issue 1: if functions have **unbounded** support, i.e. nonzero everywhere, the matrix will always be dense
  - Expensive to solve when *n* increase…
- Issue 2: the whole surface is influenced by each single input point

Bounded RBD [Morse01]

$$\varphi(d) = \begin{cases} (1-d)^p P(d), & d < 1 \\ 0, & d \geq 1 \end{cases}$$
$$P(d) = polynome \ \ with\ degree\ 6$$

- The value of *f* is determined only locally (withing the radius 1)
  - Use $\varphi(d/R)$ to adapt to the point cloud resolution
- The resulting matrix is **sparse**
- The *fitting* is local

- The … e radius 1)
  - …
- The …
- The …



8000-point model

Interpolated to 41,864 points

Bounded RBF

$$\varphi(d) = \begin{cases} (1-d)^p P(d), & d < 1 \\ 0, & d \geq 1 \end{cases}$$

$$P(d) = \textit{polynome with degree 6}$$

- The value of *f* is determined only locally (withing the radius 1)
  - Use $\varphi(d/R)$ to adapt to the point cloud resolution
- The resulting matrix is **sparse**
- The *fitting* is local

More issues:

- Still hard to represent sharp features, anisotropic basis functions may be used [Dinh01]

Partition of Unity

$$f(x) = \sum_i \varphi_i(x) Q_i\,(x)$$

- $f(x)$ is defined globally as the weighted sum of local functions that describe (implicitly) the surface
- Each i corresponds to a region of $\mathbb{R}^3$ where the function is described by $f_i(x)$
- **The sum of the weights is 1 everywhere**:

$$\sum_i \varphi_i(x) = 1$$

  - Which is obtained by normalization

$$\varphi_i(x) = \frac{\omega_i(x)}{\sum_i \omega_i(x)} \qquad \{\omega_i(\mathbf{x})\}\ s.t.\ \Omega \subset \bigcup_i supp(\omega_i)$$

Multilevel PoUI [Ohtake03]

- Starting from the bounding box of the point cloud, build an **octree**
- The rule for creating the children of a node is:
  Can we define an implicit surface with the point corresponding to the cell as:

$$f(x) = \sum_i \varphi_i(x) Q_i\ (x)$$

  – for $Q_i\ (x)$ in a set of predefined shape functions
  – With and approsimation error less than ε ?

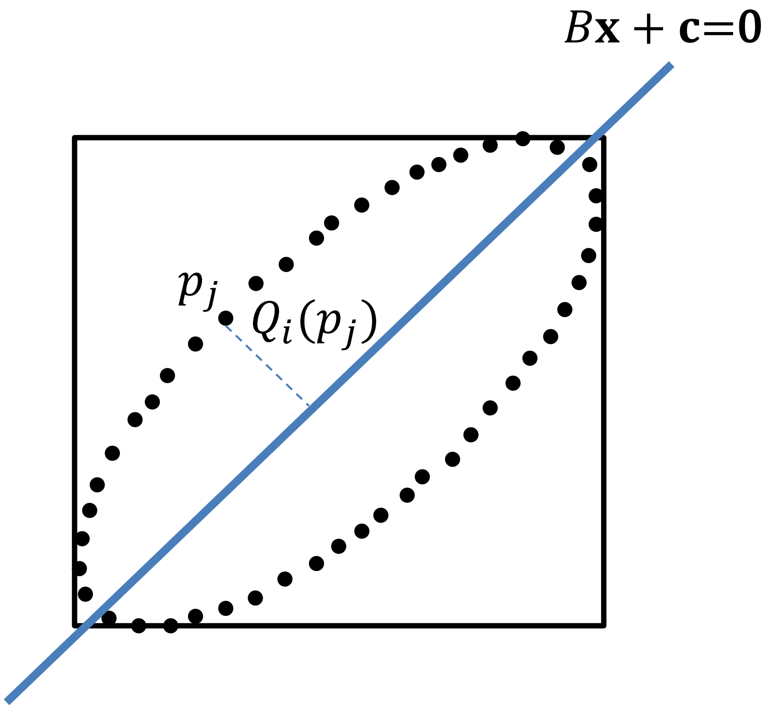
Multilevel PoUI [Ohtake03]

- (simplified) Example

$$f(x) = \sum_i \varphi_i(x) Q_i\ (x)$$

shape $Q_i(x) = B\mathbf{x} + \mathbf{c}$

approx $\varepsilon = \sum_j |Q_i(p_j)|$



Error is big, split

Multilevel PoUI [Ohtake03]

- (simplified) Example

$$f(x) = \sum_i \varphi_i(x) Q_i\ (x)$$

shape $\quad Q_i(x) = B\mathbf{x} + \mathbf{c}$

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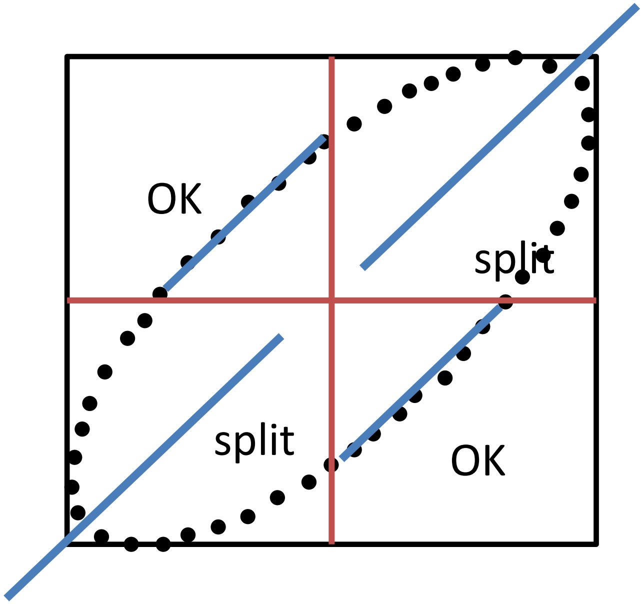

Multilevel PoUI [Ohtake03]

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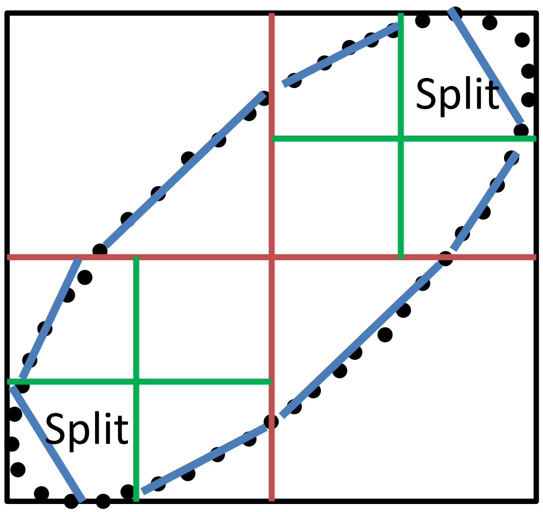

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Multilevel PoUI [Ohtake03]

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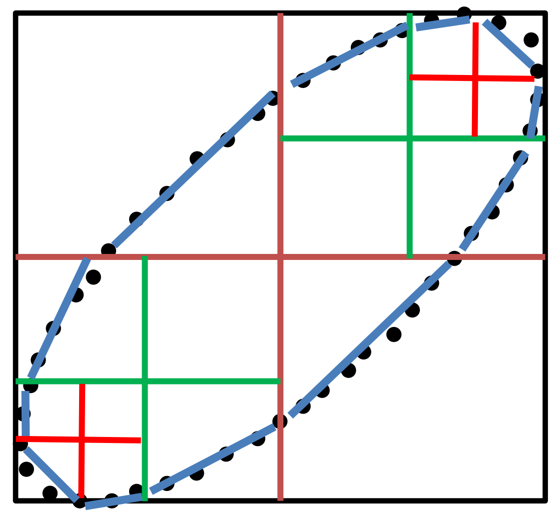
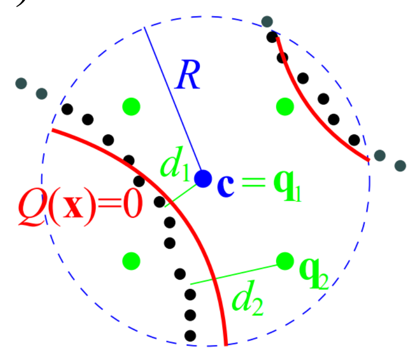


Multilevel PoUI

- Subdivide the domain with an **octree**
- Fit the points within each cell with a function $Q_i(x)$, either:
  - A quadric (for noisy and unbounded regions)

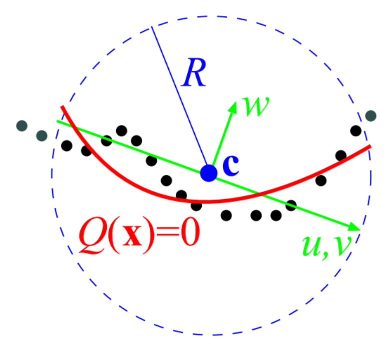
$$Q_i\,(\mathbf{x}) = \mathbf{x}^{\mathbf{T}}\mathbf{A}\mathbf{x} + \mathbf{b}^{\mathbf{T}}\mathbf{x} + \mathbf{c}$$



  - A bivariate (u,v) quadratic polynomial in a local coordinate system (for smooth patch)

$$Q_i\,(\mathbf{x}) = w - [u,v]^{\mathbf{T}}\mathbf{A}\begin{bmatrix}u\\v\end{bmatrix} + \mathbf{b}^{\mathbf{T}}\begin{bmatrix}u\\v\end{bmatrix} + \mathbf{c}$$

$[u,v,w]^T$ point expressed in a local frame

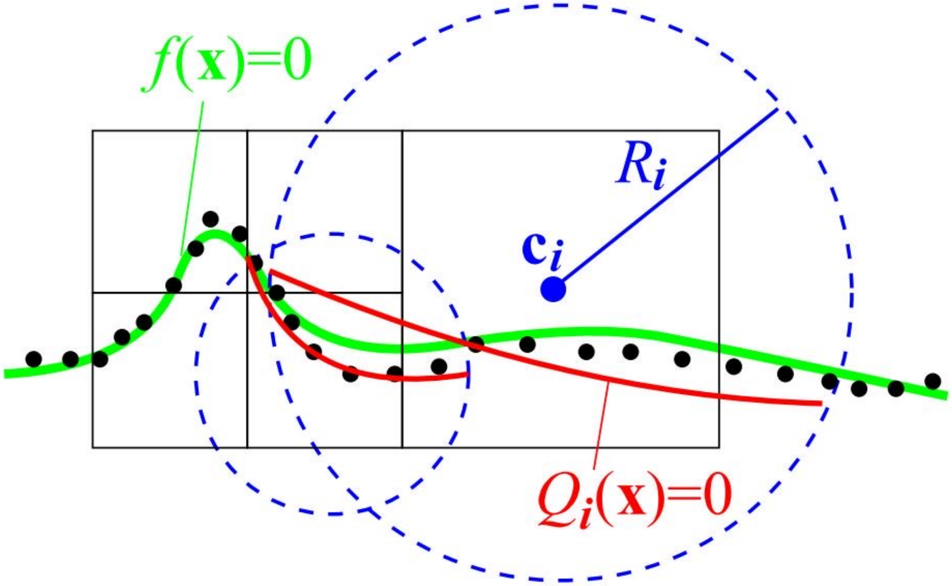


Multilevel PoUI

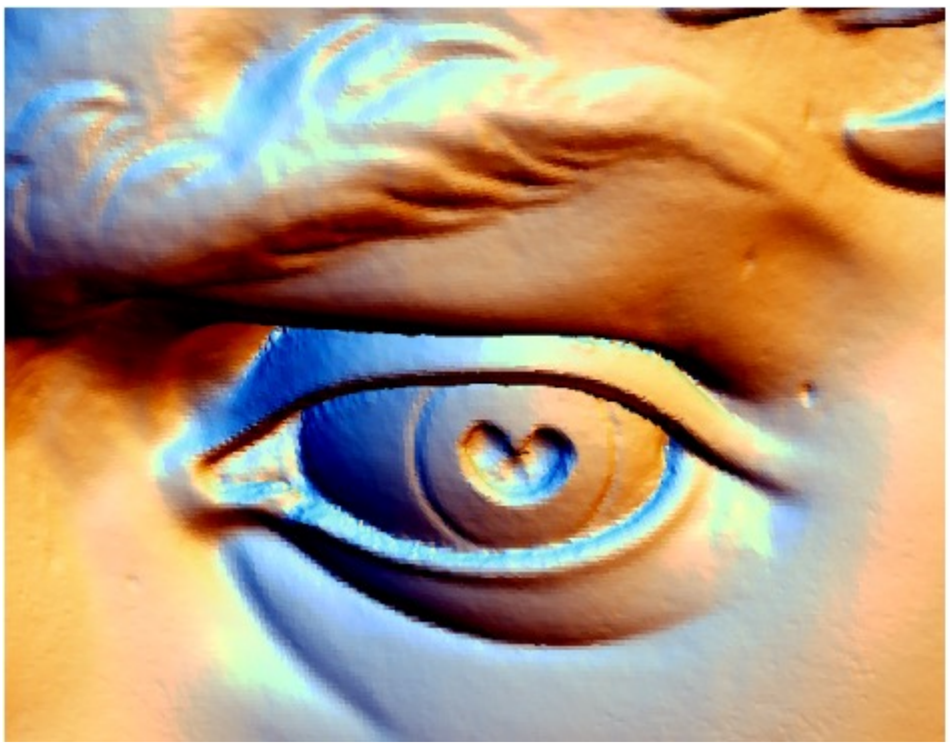
- Subdivide the domain with an **octree**
- Fit the points within each cell with a function $Q_i(x)$, either:
  - A quadric (for noisy and unbounded regions)
  - A bivariate (u,v) quadratic polynomial in a local coordinate system (for smooth patch)
  - A piecewise quadratic surface (for sharp features)
- Blending PU:

$$\omega_i(x) = b\left(\frac{3|x - c_i|}{2R_i}\right)$$

$$R_i = 0.75 * diag$$



Results



Distance field from range maps [Levoy]

MPU implicits

Moving Least Square Reconstruction

LS
Least square

$$\min_{f\in\prod_m^d} \sum_i \|f(x_i) - f_i\|$$

$\prod_m^d$:polynomes degree m in d-dimension

WLS
Weighted
Least square

$$\min_{f_{\overline{x}}\in\prod_m^d} \sum_i \theta(\|x_i - \overline{x}\|)\|f(x_i) - f_i\|$$

$\bar{x}$: fixed point

MLS
Moving
Least square

$$\min_{f_x\in\prod_m^d} \sum_i \theta(\|x_i - x\|) \ \ \|f_x(x_i) - f_i\|$$

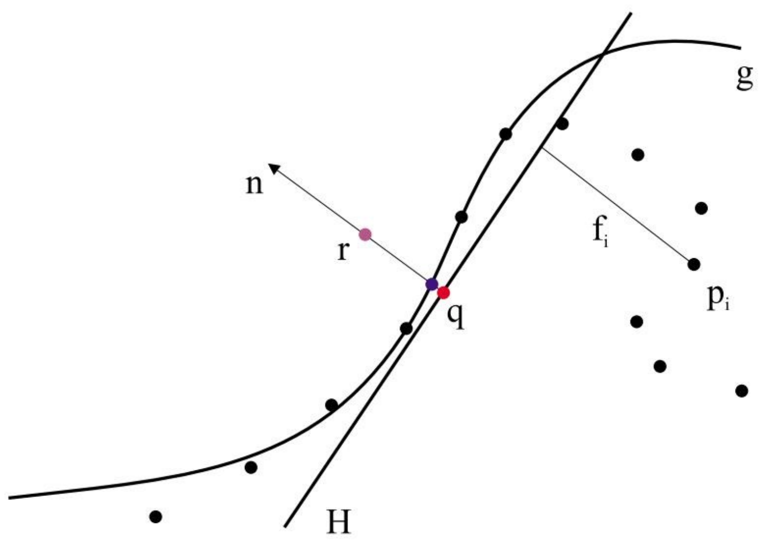
Moving Least Square Reconstruction
[Alexa01]

- Iterative approach: project the points near the surface onto the surface (??)

1. $\min_{n,t} \sum_{i=1}^{N} \langle n, p_i - r - tn \rangle^2 \, \theta(\|p_i - r - tn\|)$

2. $\min_g \sum_{i=1}^{N} (g(x_i, y_i) - f)^2 \, \theta(\|p_i - q\|)$

3. Move r to $q + g(0,0)\, n$



Moving Least Square Reconstruction

[Alexa01]

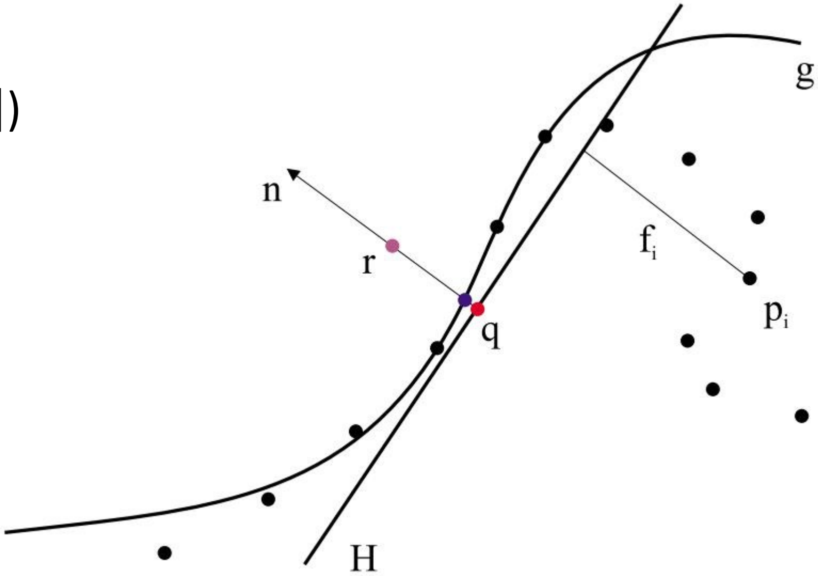
- Iterative approach: project the points near the surface onto the surface (??)

Squared distance between p_i and the plane n, t

1. $\min_{n,t} \sum_{i=1}^{N} \boxed{\langle n, p_i - r - tn \rangle^2} \theta(\|p_i - r - tn\|)$ Non linear problem

2. $\min_{g} \sum_{i=1}^{N} (g(x_i, y_i) - f)^2 \, \theta(\|p_i - q\|)$

3. Move r to $q + g(0,0)\, n$



Moving Least Square Reconstruction

[Alexa01]

- Iterative approach: project the points near the surface onto the surface

1. $$\min_{n,t} \sum_{i=1}^{N} \langle n, p_i - r - tn \rangle^2 \, \theta(\|p_i - r - tn\|)$$

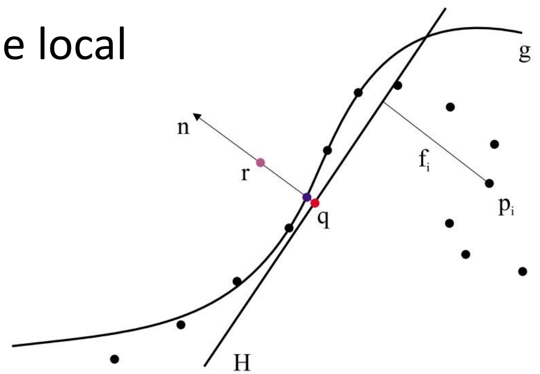
$$f_i = \boxed{n} \cdot (p_i - q)$$

Known from 1.

2. $$\min_g \sum_{i=1}^{N} \boxed{(g(x_i, y_i) - f_i)^2} \, \theta(\|p_i - \boxed{q}\|)$$

Non linear problem

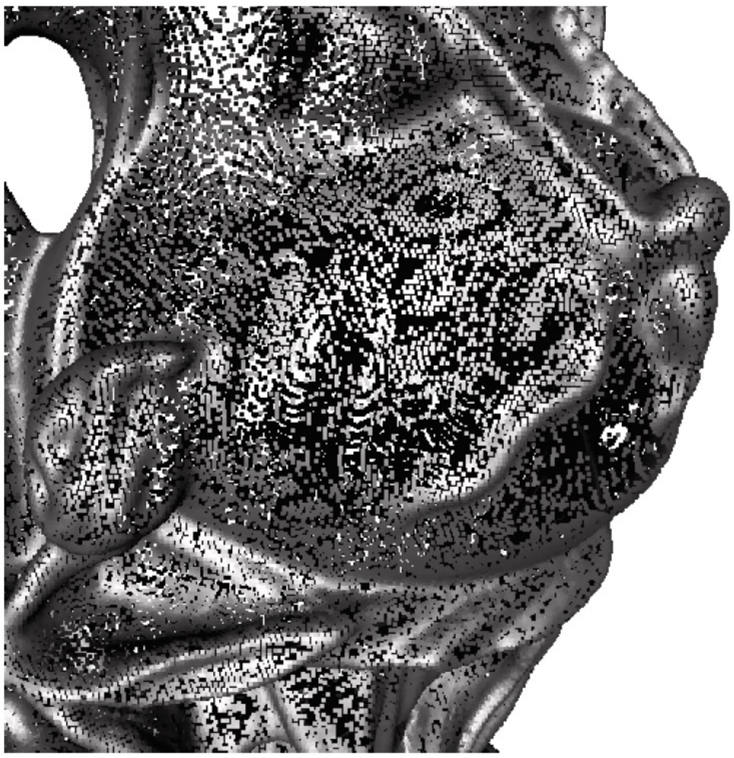
$g: \mathbb{R}^2 \Longrightarrow \mathbb{R}$ approximates point set in the local reference system centered in q



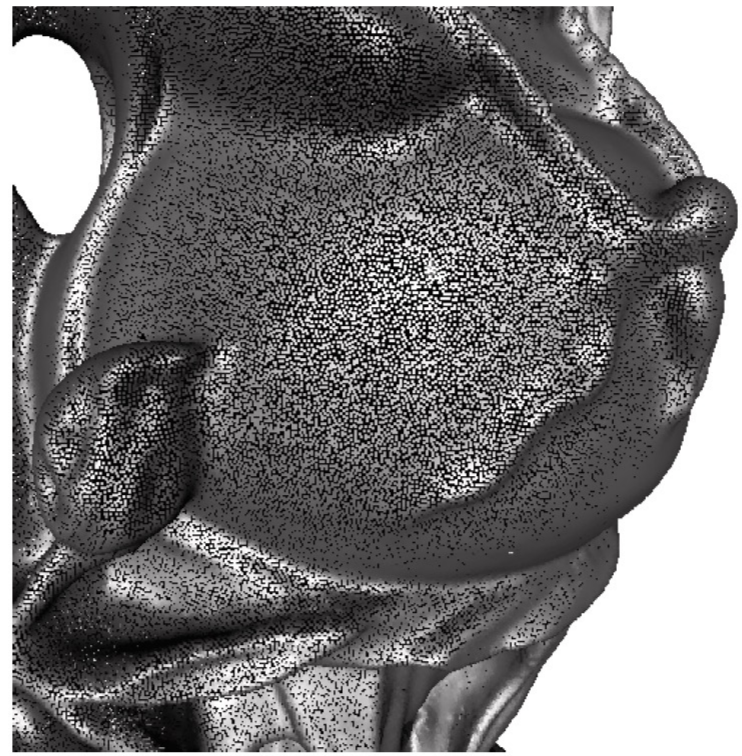
3. Move r to $q + g(0,0)\, n$

- Repeat 1-3 until stationary point (r projects on itself)

Moving Least Square Reconstruction



Irregular sampling as
acquired by a laser scanner



After MLS reconstruction

Moving Least Square Reconstruction

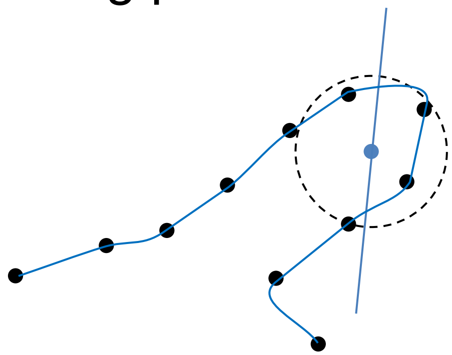
$\theta(d) = e^{-\frac{d^2}{h^2}}$ *h is related to the spacing between samples*

Smaller *h*

Larger *h*

Algebraic Point Set Surfaces [Guennebaud07]

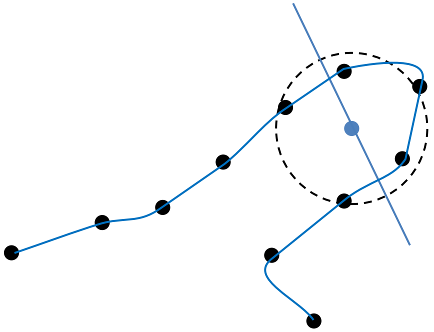
- Plane fitting problem with MLS:



  - Nearby position lead to very different planes estimation
  - Opposite sheets of surface considered as one

Algebraic Point Set Surfaces [Guennebaud07]

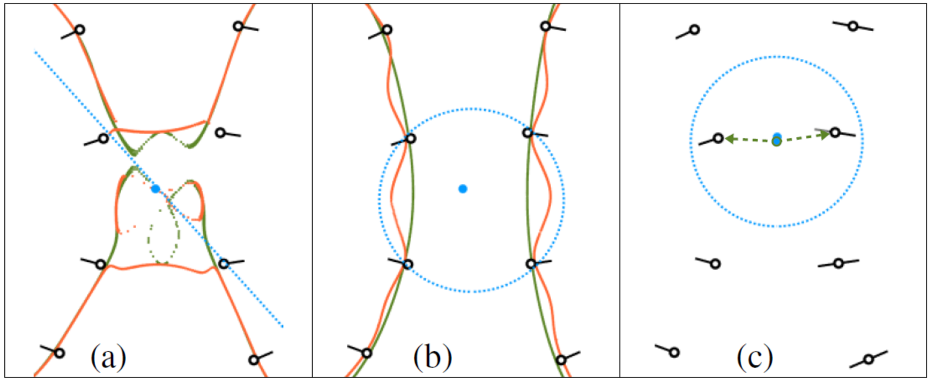
- Plane fitting problem with MLS:



  - Nearby position lead to very different planes estimation
  - Opposite sheets of surface considered as one

Algebraic Point Set Surfaces [Guennebaud07]

- **Main idea**: fit *spheres* insted of planes
  - Spheres to define normal at the points
  - Spheres to define the surface in the MLS iteration



Plane fitting   Sphere fitting

Algebraic Point Set Surfaces [Guennebaud07]

- Sphere fitting
  - Geometric fitting is unstable for planar configuration
  - Use an algebraic approach, define the surface of the sphere as the zeroes of the function $S_{\mathbf{u}}(\mathbf{x})$:

$$S_{\mathbf{u}}(\mathbf{x}) = [1, \mathbf{x}^{\mathbf{T}}, \mathbf{x}^{\mathbf{T}}\mathbf{x}]\,\mathbf{u}, \quad \mathbf{u} = [\mathrm{u}_0, \dots, \mathrm{u}_{\mathrm{d}+1}]$$

$$S_{\mathbf{u}}(\mathbf{x}) = u_0 + u_1 x + u_2 y + u_3 z + u_4(x^2 + y^2 + z^2)$$

center $\mathbf{c} = -\frac{1}{2u_4}[u_1, u_2, u_3]^T$

radius $r = \sqrt{\mathbf{c}^{\mathbf{T}}\mathbf{c} - u_0/u_4}$

  - $u_4 = 0 \rightarrow S_{\mathbf{u}}(\mathbf{x}) = 0$ defines a plane

Algebraic Point Set Surfaces [Guennebaud07]

$$\mathbf{W}(\mathbf{x}) = \begin{bmatrix} w_0(\mathbf{x}) & & \\ & \ddots & \\ & & w_{n-1}(\mathbf{x}) \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 1 & \mathbf{p}_0^T & \mathbf{p}_0^T\mathbf{p}_0 \\ \vdots & \vdots & \vdots \\ 1 & \mathbf{p}_{n-1}^T & \mathbf{p}_{n-1}^T\mathbf{p}_{n-1} \end{bmatrix}.$$

Algebraic sphere fitting in a neighborhood of *n* points

$$\boxed{\mathbf{u}(\mathbf{x}) = \underset{\mathbf{u},\,\mathbf{u}\neq\mathbf{0}}{\arg\min} \left\| \mathbf{W}^{\frac{1}{2}}(\mathbf{x})\mathbf{D}\mathbf{u} \right\|^2}$$

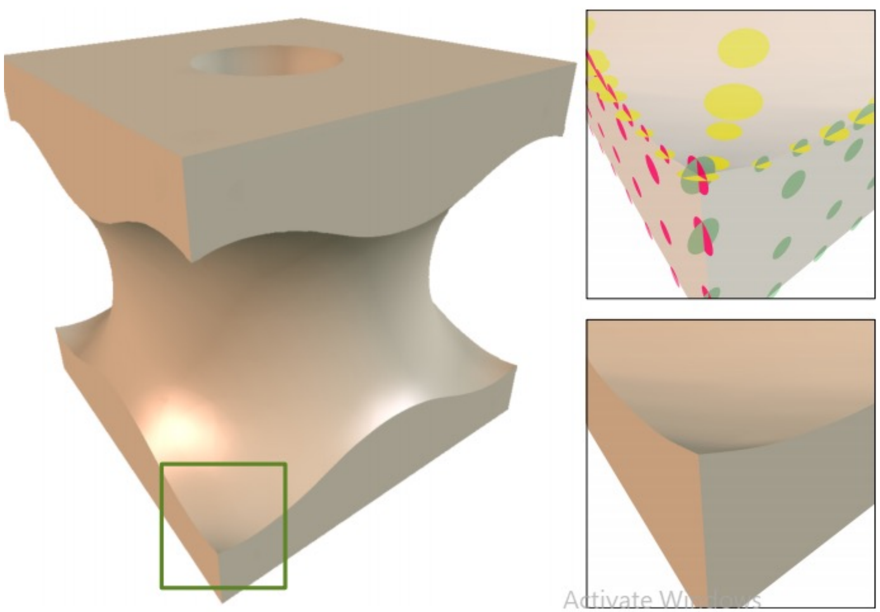
Weighting scheme

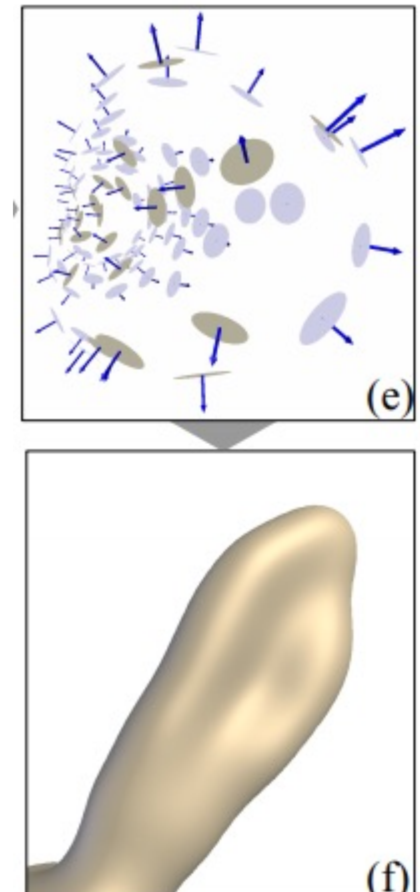
$$w_i(\mathbf{x}) = \phi\left(\frac{\|\mathbf{p}_i - \mathbf{x}\|}{h_i(\mathbf{x})}\right)$$

Sampling radii

$$\phi(x) = \begin{cases} (1-x^2)^4 & \text{if } x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Algebraic Point Set Surfaces [Guennebaud07]

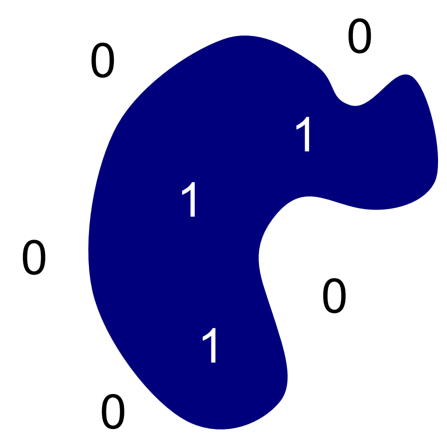




Poisson surface reconstruction

- We reconstruct the surface of the model by solving for the indicator function of the shape.

$$\chi_M(p) = \begin{cases} 1 & \text{if } p \in M \\ 0 & \text{if } p \notin M \end{cases}$$



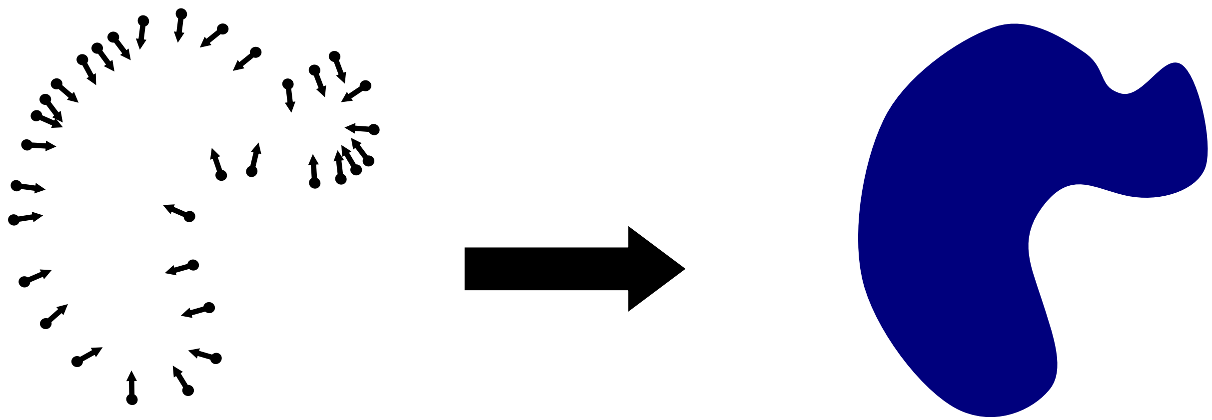
Indicator function

χ_M

[credits: from the original Huges Hoppe slides]

Challenge

- How to construct the indicator function?



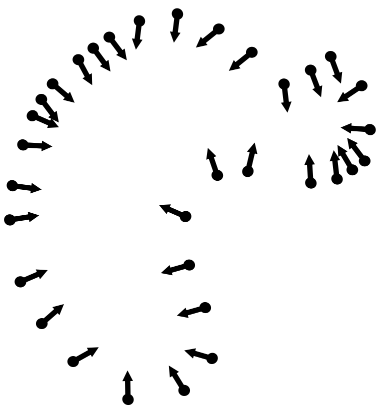
Oriented points

Indicator function

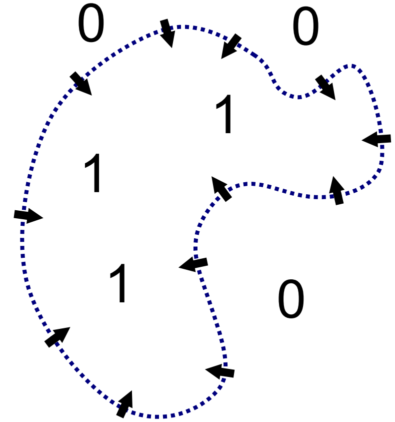
χ_M

Gradient Relationship

- There is a relationship between the normal field and gradient of indicator function



Oriented points



Indicator gradient

$\nabla \chi_M$

Integration

- Represent the normals by a vector field $\vec{V}$
- Find the function χ whose gradient best approximates $\vec{V}$:

$$\min_{\chi} \left\| \nabla \chi - \vec{V} \right\|$$

Integration as a Poisson Problem

- Represent the points by a vector field $\vec{V}$
- Find the function χ whose gradient best approximates $\vec{V}$:

$$\min_{\chi} \left\| \nabla \chi - \vec{V} \right\|$$

- Applying the divergence operator, we can transform this into a Poisson problem:

$$\nabla \cdot (\nabla \chi) = \nabla \cdot \vec{V} \quad \Leftrightarrow \quad \Delta \chi = \nabla \cdot \vec{V}$$

Vector field approximation from samples

- Note: the indicator function is discontinuous, how can we compute its gradient?
- Smoothing Filter:

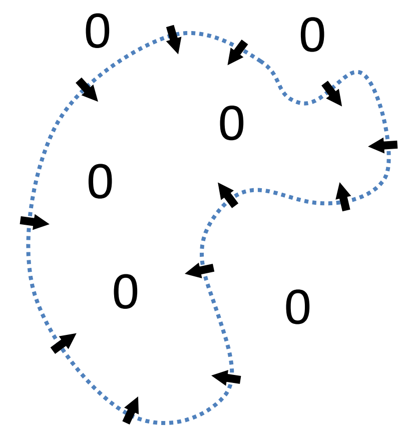
Lemma: [kazhdan06]

M manifold, $\vec{N}_{\partial M}(p)$ surface normal,

$\tilde{F}$ smoothing filter:

$$\tilde{F}_p(q) = \tilde{F}(q - p)$$

$$\nabla\left(\chi_M * \tilde{F}\right)(q_0) = \int_{\partial M} \tilde{F}_p(q_0)\vec{N}_{\partial M}(p)\, dp$$



Indicator gradient

$\nabla \chi_M$

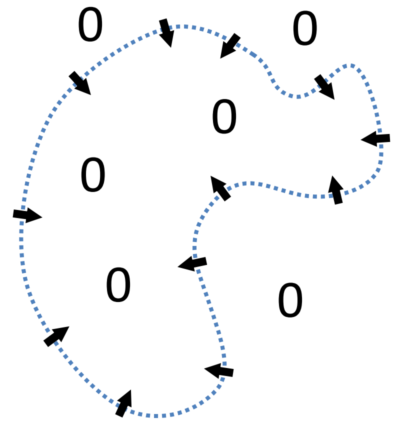
Vector field approximation from samples

- Note: the indicator function is discontinuous, how can we compute its gradient?
- Smoothing Filter:

$$\nabla\left(\chi_M * \tilde{F}\right)(q_0) = \int_{\partial M} \tilde{F}_p(q_0)\vec{N}_{\partial M}(p)\,dp =$$

$$\sum_{s\in S}\int_{\mathcal{P}_s} \tilde{F}_p(q)\,\vec{N}_{\partial M}(p)dp \approx$$

$$\sum_{s\in S}|\mathcal{P}_s|\tilde{F}_{s.p}(q)\,s.\vec{N}\,dp \equiv \vec{V}$$



Indicator gradient

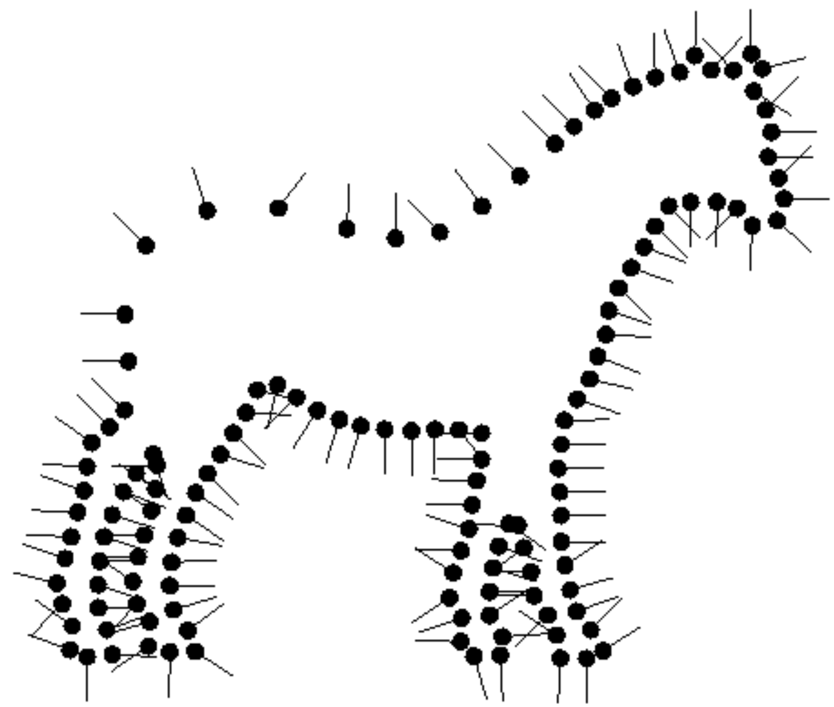
$\nabla\chi_M$

[credits: from the original Huges Hoppe slides]

Implementation

Given the Points:

- Set octree
- Compute vector field
- Compute indicator functior
- Extract iso-surface

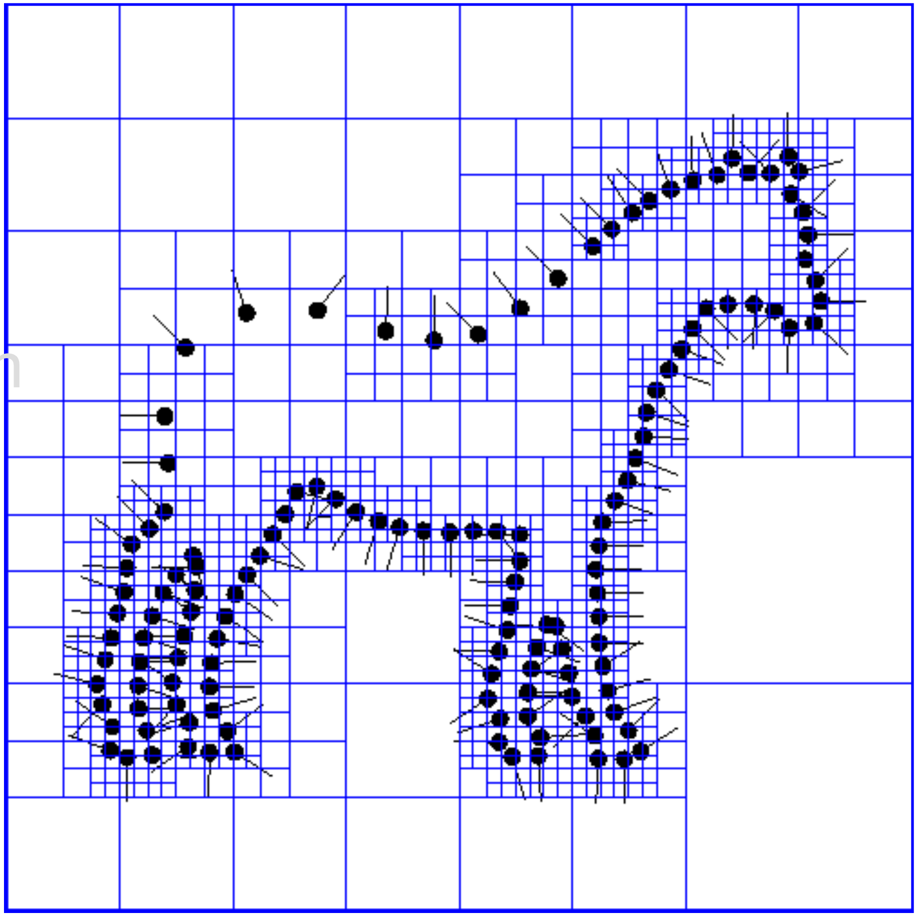


[credits: from the original Huges Hoppe slides]

Implementation: Adapted Octree

Given the Points:

- Set octree
- Compute vector field
- Compute indicator function
- Extract iso-surface

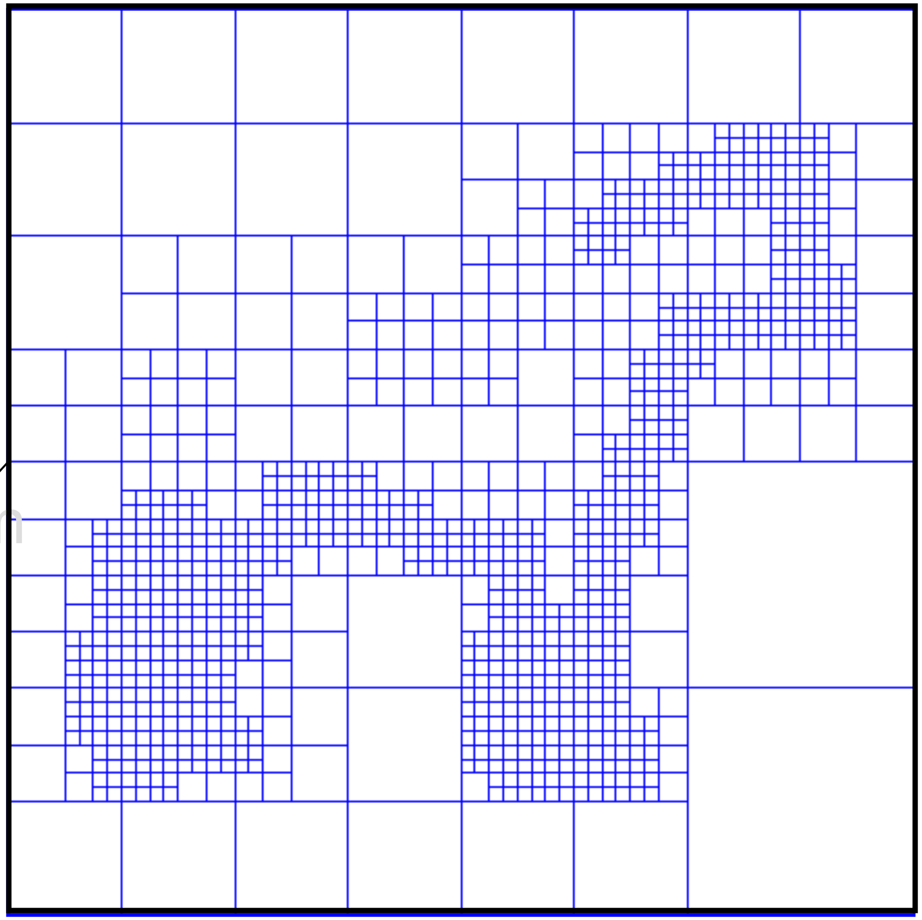


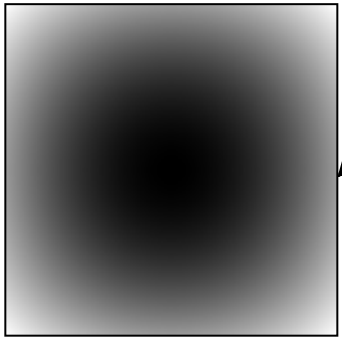
[credits: from the original Huges Hoppe slides]

Implementation: Vector Field

Given the Points:

- Set octree
- Compute vector field
  - Define a function space
  - Splat the samples
- Compute indicator function
- Extract iso-surface





[credits: from the original Huges Hoppe slides]

Implementation: Vector Field

Given the Points:

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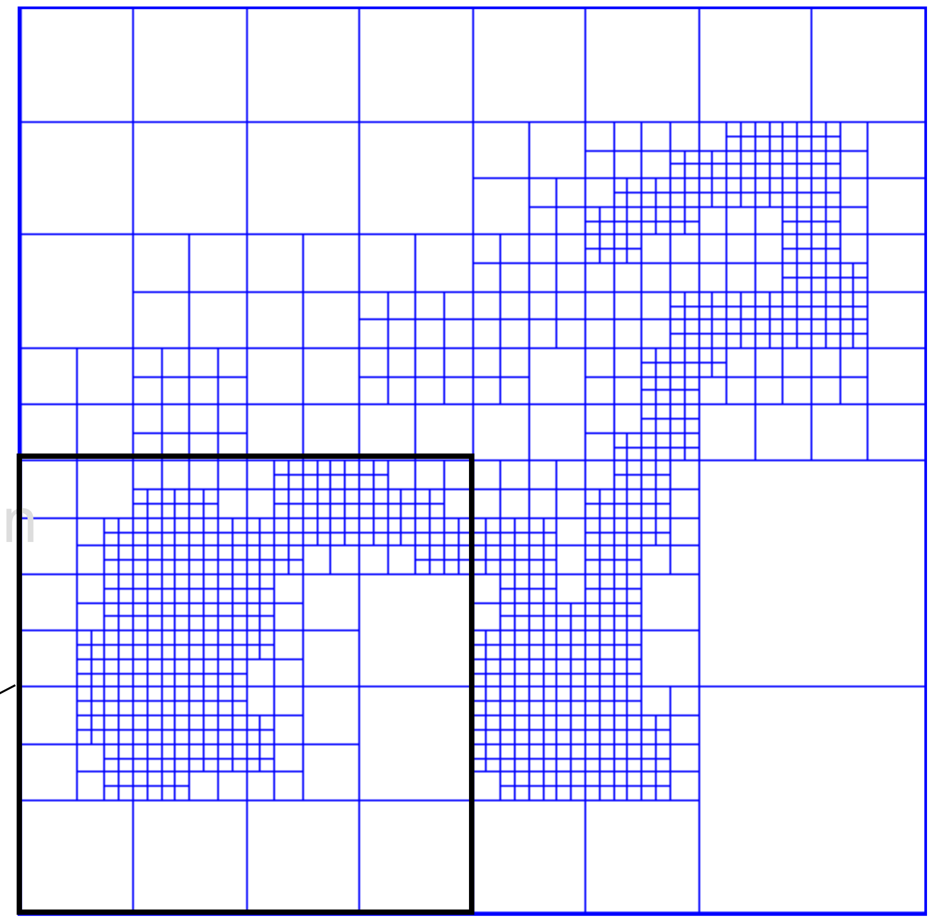

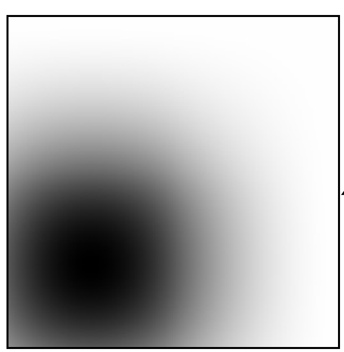

Implementation: Vector Field

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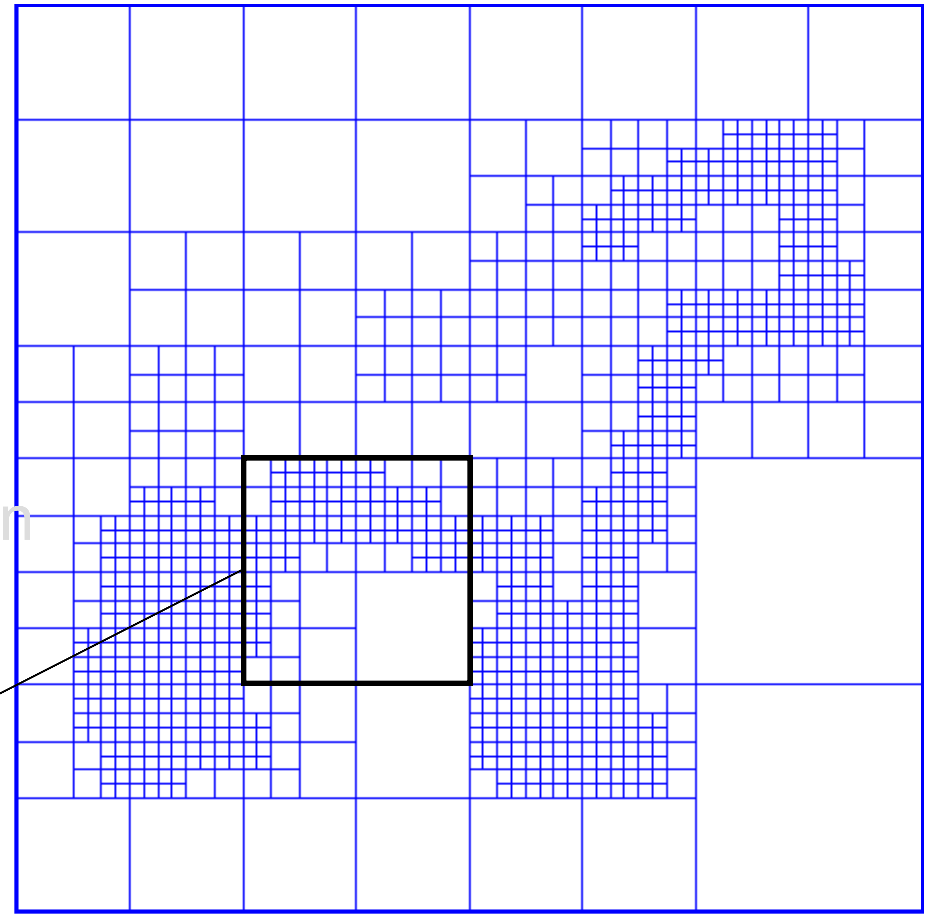

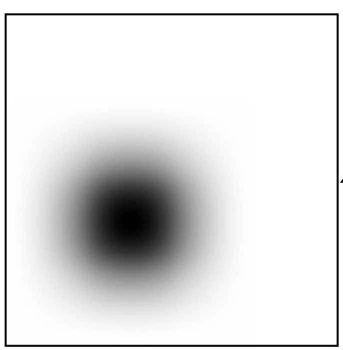

[credits: from the original Huges Hoppe slides]

Implementation: Vector Field

Given the Points:

- Set octree
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  - Define a function space
  - Splat the samples
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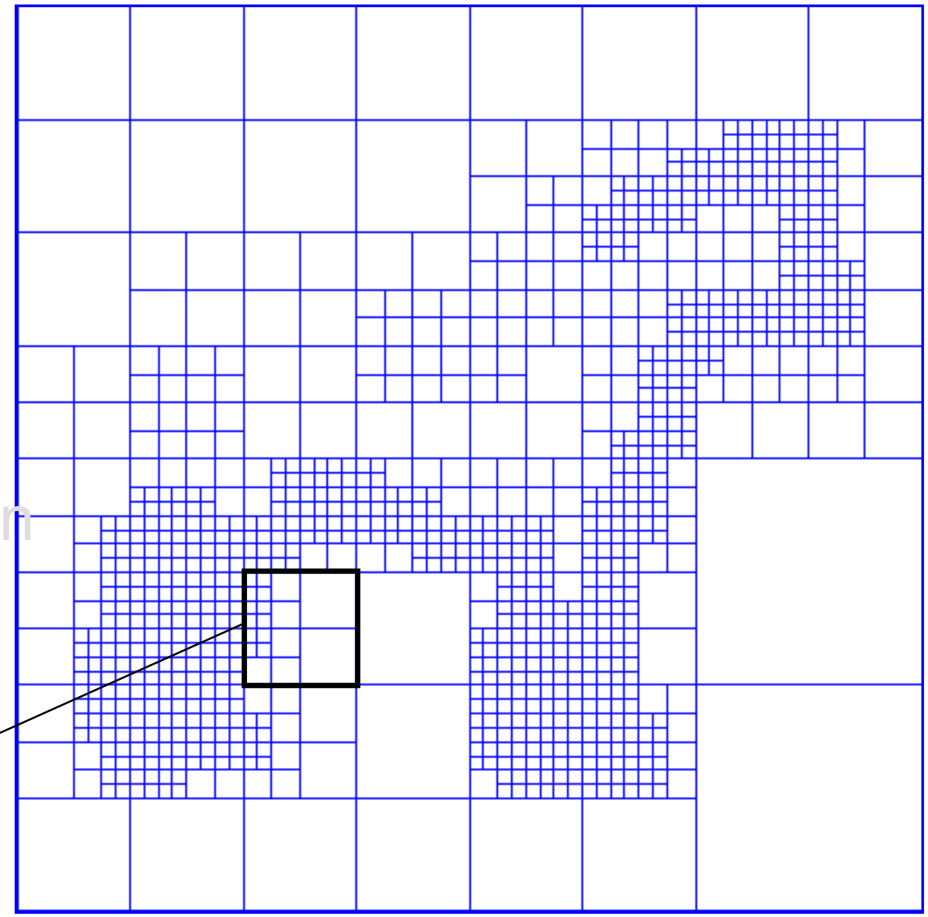
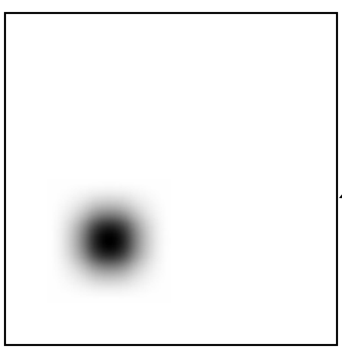
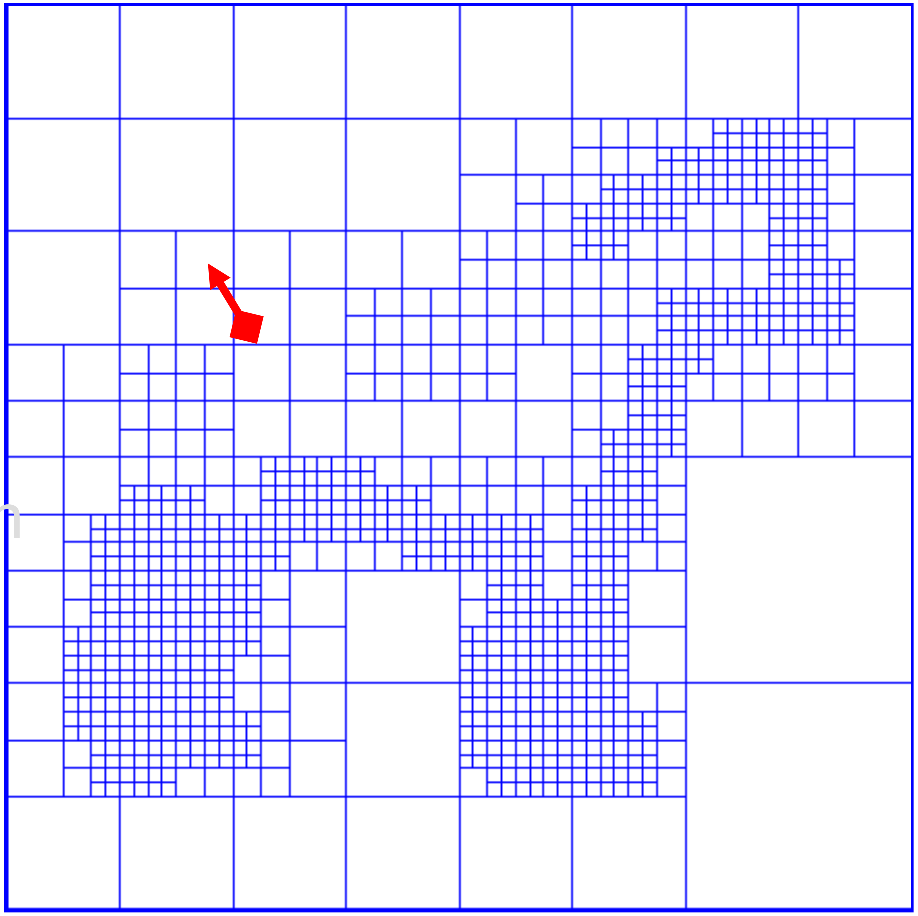




Implementation: Vector Field

Given the Points:

- Set octree
- Compute vector field
  - Define a function basis
  - Splat the samples
- Compute indicator function
- Extract iso-surface



[credits: from the original Huges Hoppe slides]

Implementation: Vector Field

Given the Points:

- Set octree
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  - Define a function basis
  - Splat the samples
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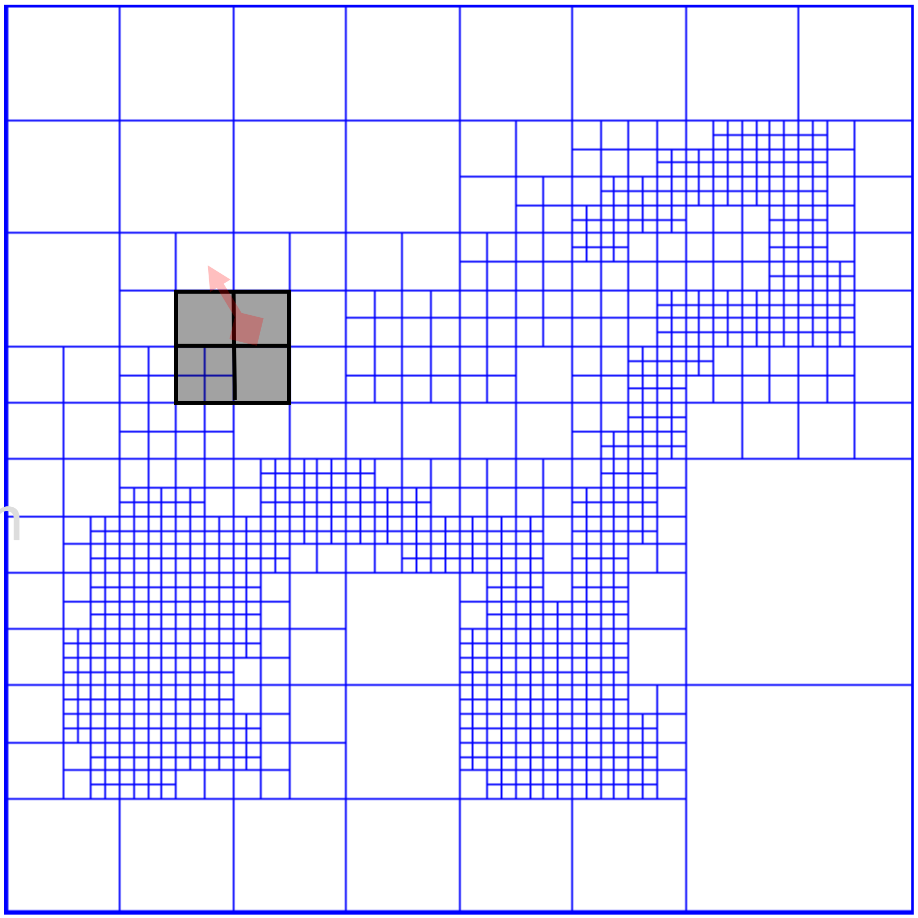
- Extract iso-surface



[credits: from the original Huges Hoppe slides]

Implementation: Vector Field

Given the Points:

- Set octree
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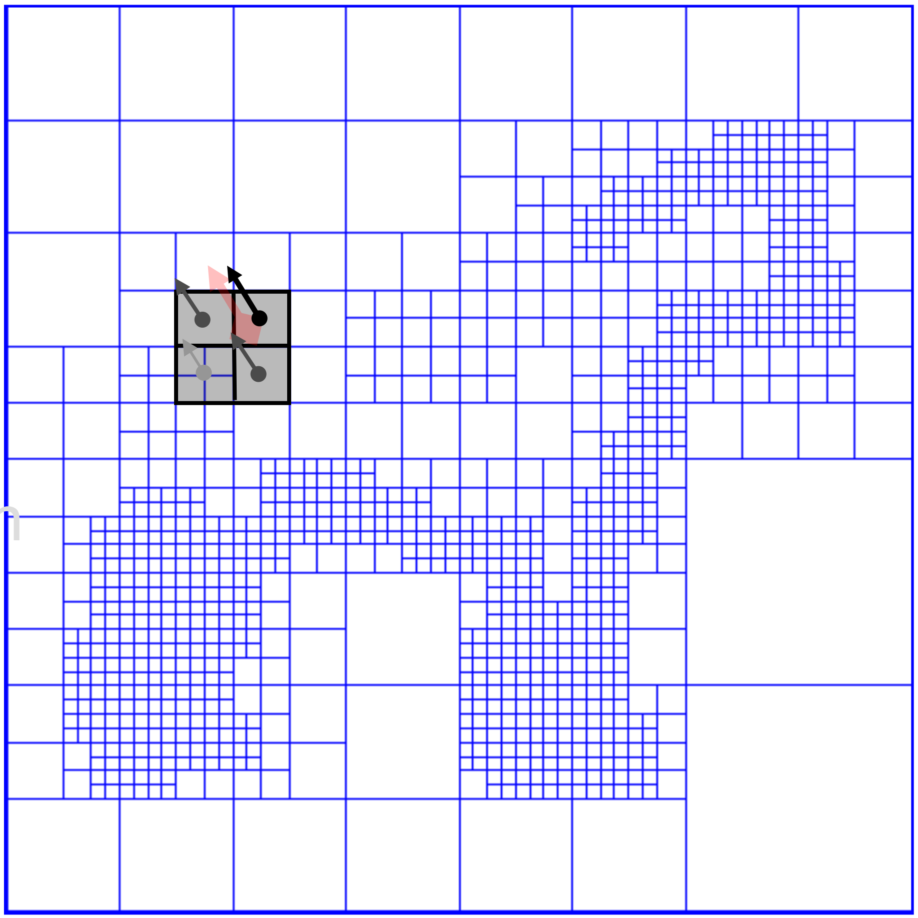
- Compute indicator function
- Extract iso-surface

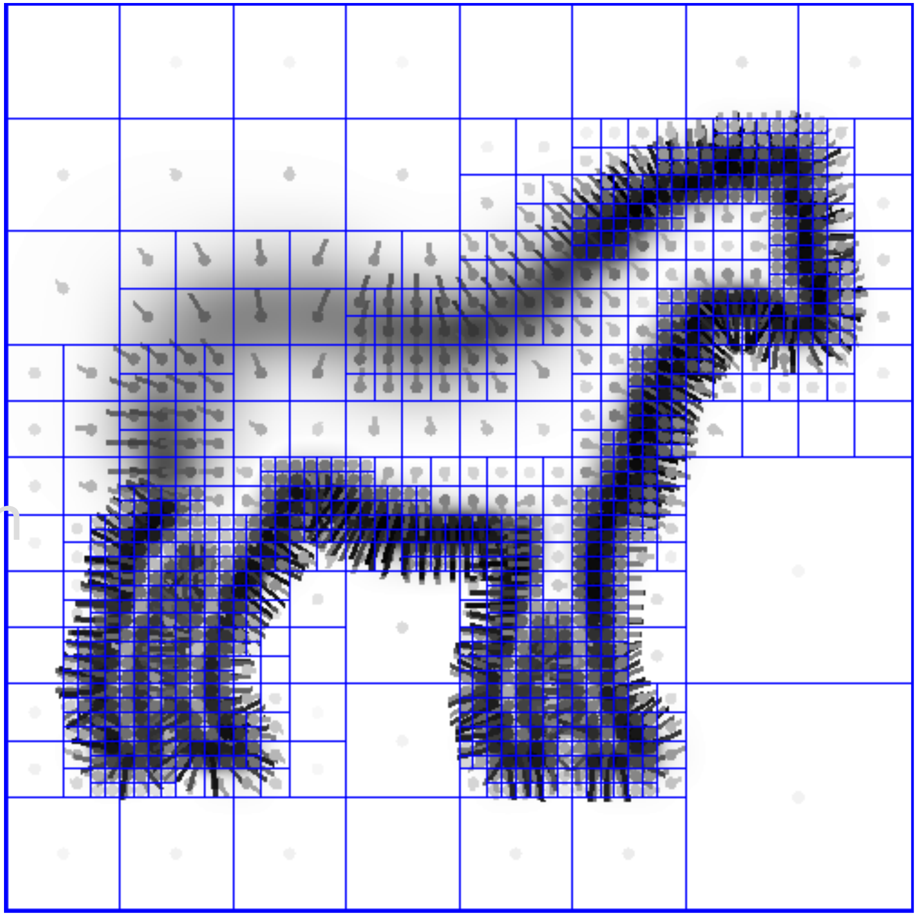


[credits: from the original Huges Hoppe slides]

Implementation: Vector Field

Given the Points:

- Set octree
- Compute vector field
  - Define a function space
  - Splat the samples
- Compute indicator function
- Extract iso-surface



Setting up the minimization problem

- So we have defined the vector field

$$\sum_{s \in S} |\mathcal{P}_s| \tilde{F}_{s.p}(q)\, s.\vec{N}\, dp \equiv \vec{V}$$

- ...can't we just integrate it and get χ?
  - No, no guarantees that $\vec{V}$ is curl free

Setting up the minimization problem

- Minimize $|\Delta\chi - \nabla \vec{V}|$ instead...
- ... More precisely minimize the difference of their projections on the basis of functions F

$$\sum_{o\in\mathcal{O}} \left\| \langle \Delta\tilde{\chi} - \nabla\cdot\vec{V}, F_o\rangle \right\|^2 = \sum_{o\in\mathcal{O}} \left\| \langle \Delta \boxed{\tilde{\chi}}, F_o\rangle - \langle \nabla\cdot\vec{V}, F_o\rangle \right\|^2$$

The unknown

All the nodes of
The octree

$$\min_{x\in\mathbb{R}^{|\mathcal{O}|}} \|L\boxed{x} - v\|^2$$

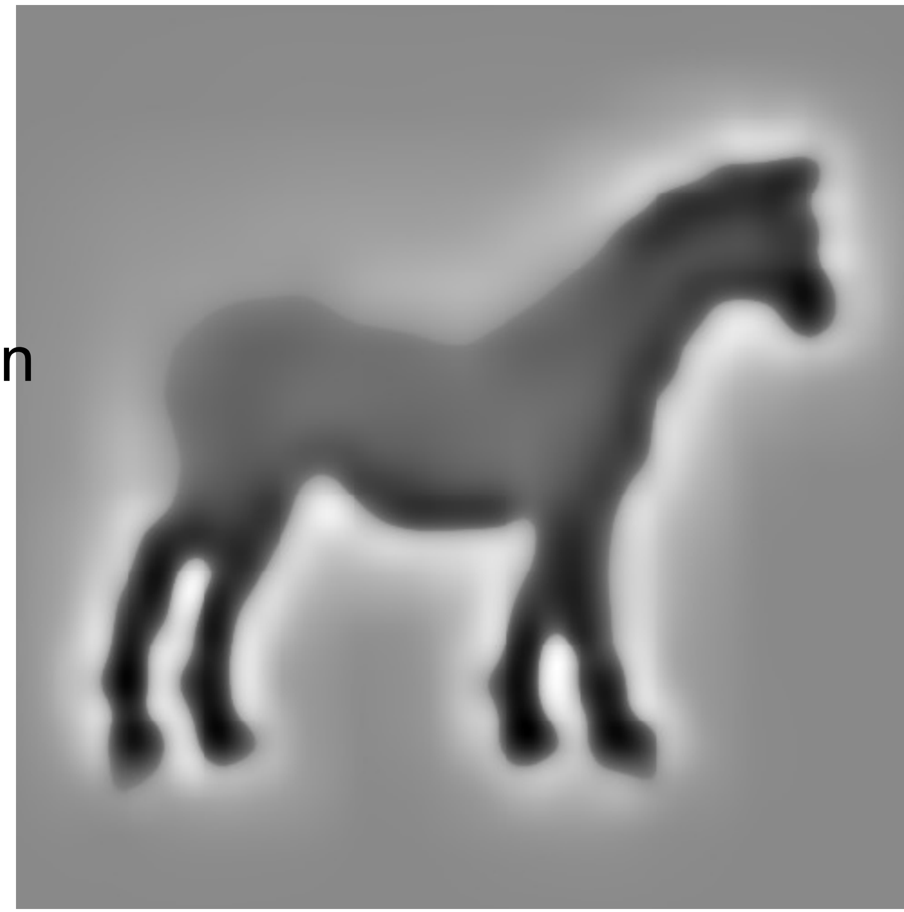
Coefficients producing χ

[credits: from the original Huges Hoppe slides]

Implementation: Indicator Function

Given the Points:

- Set octree
- Compute vector field
- Compute indicator function
  - Compute divergence
  - Solve Poisson equation
- Extract iso-surface

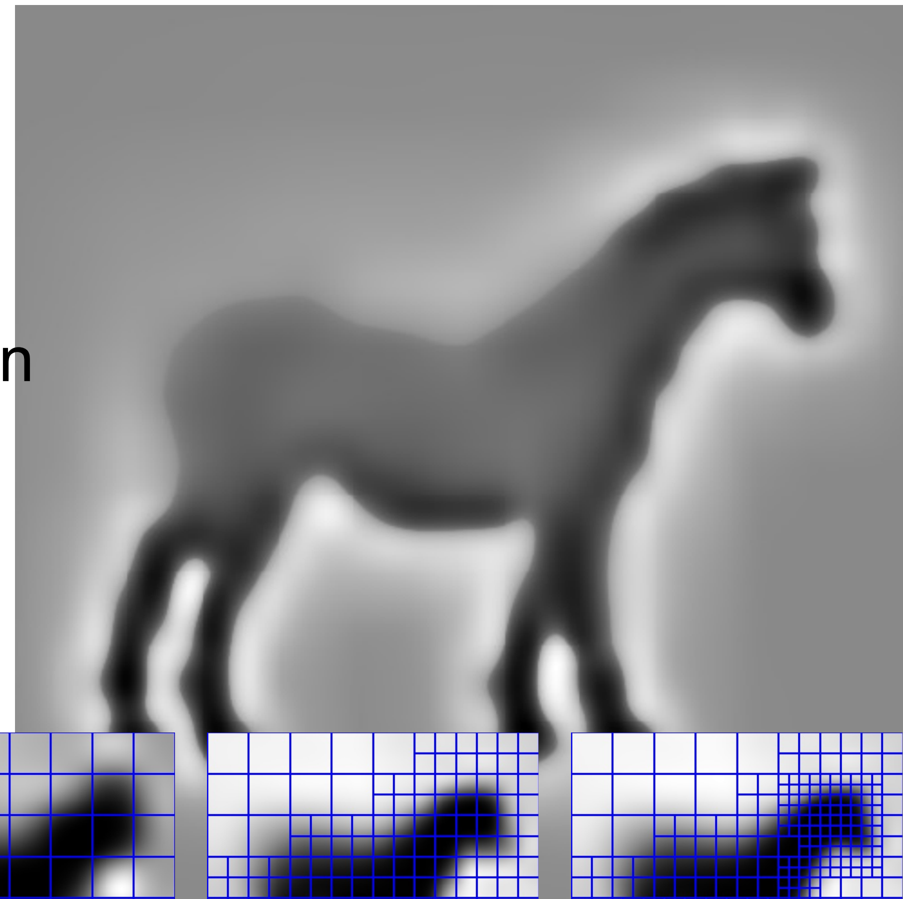


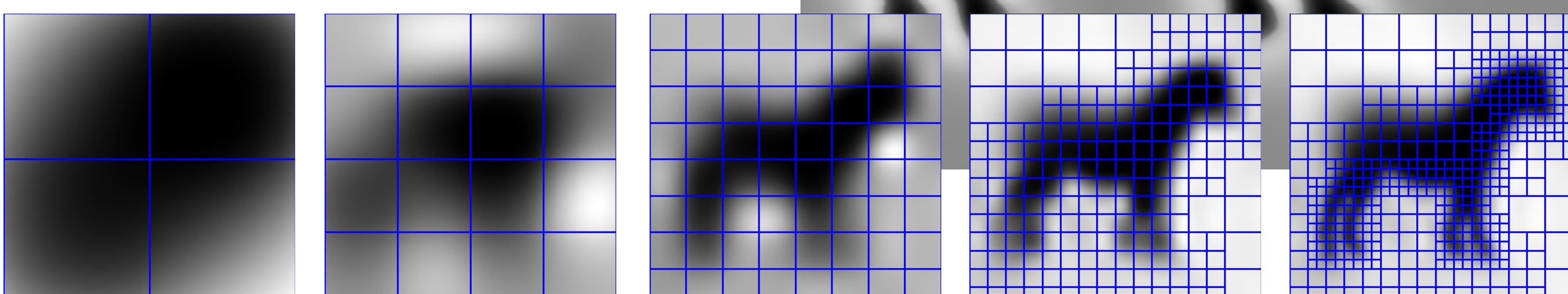
[credits: from the original Huges Hoppe slides]

Implementation: Indicator Function

Given the Points:

- Set octree
- Compute vector field
- Compute indicator function
  - Compute divergence
  - Solve Poisson equation
- Extract iso-surface

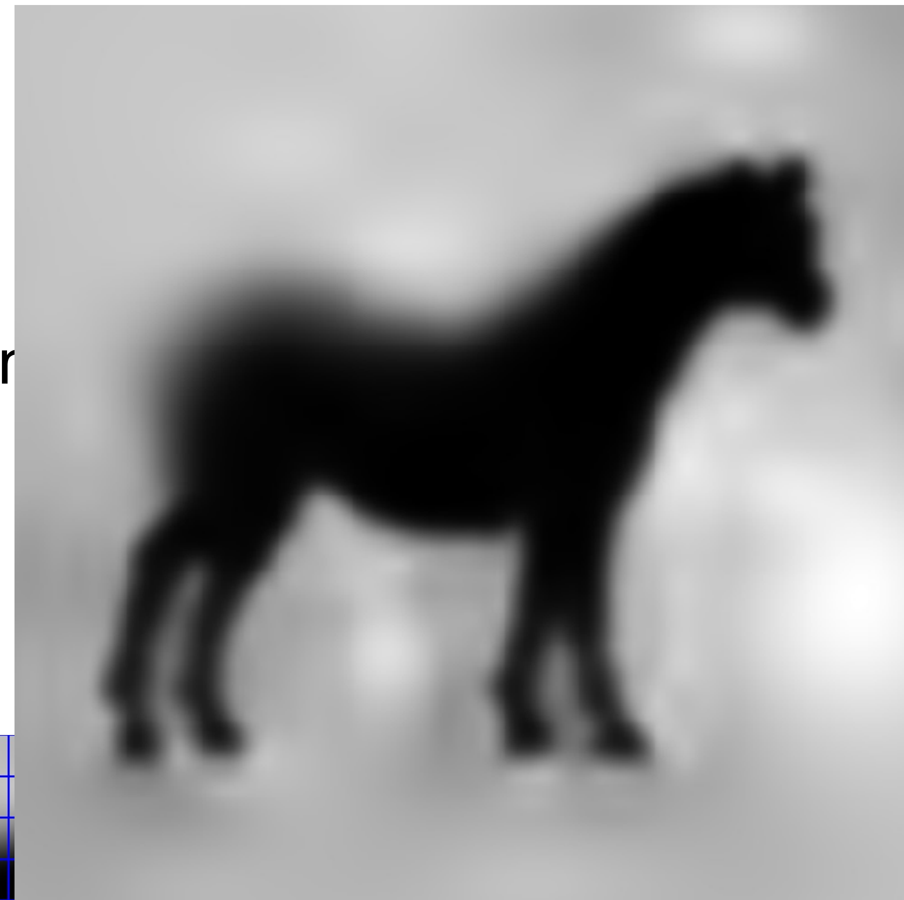


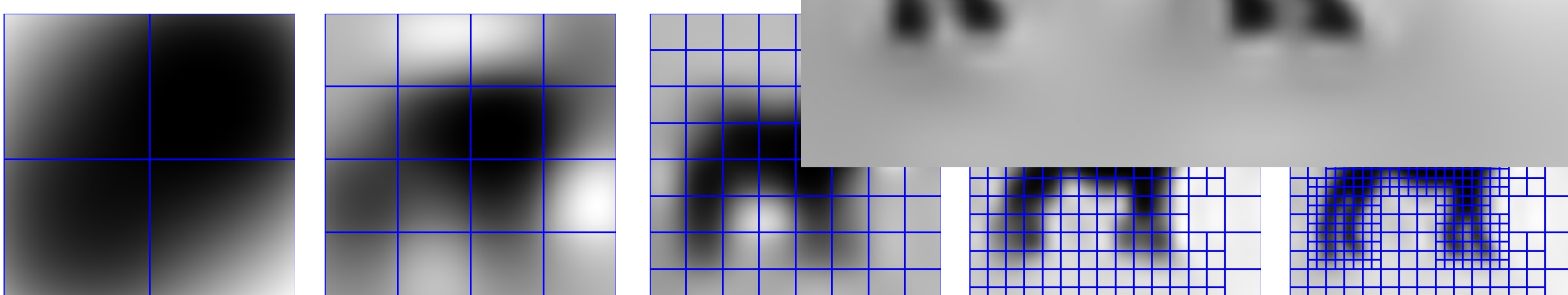


Implementation: Indicator Function

Given the Points:

- Set octree
- Compute vector field
- Compute indicator functio
  - Compute divergence
  - Solve Poisson equation
- Extract iso-surface



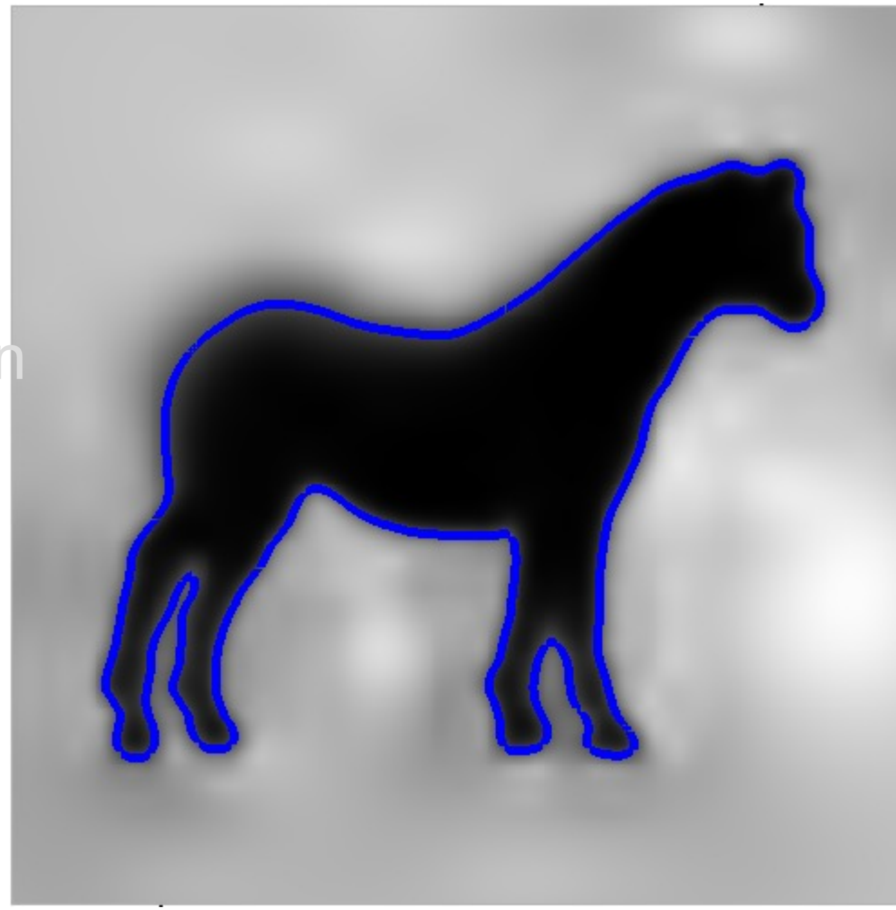


[credits: from the original Huges Hoppe slides]

Implementation: Surface Extraction

Given the Points:

- Set octree
- Compute vector field
- Compute indicator function
- Extract iso-surface



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