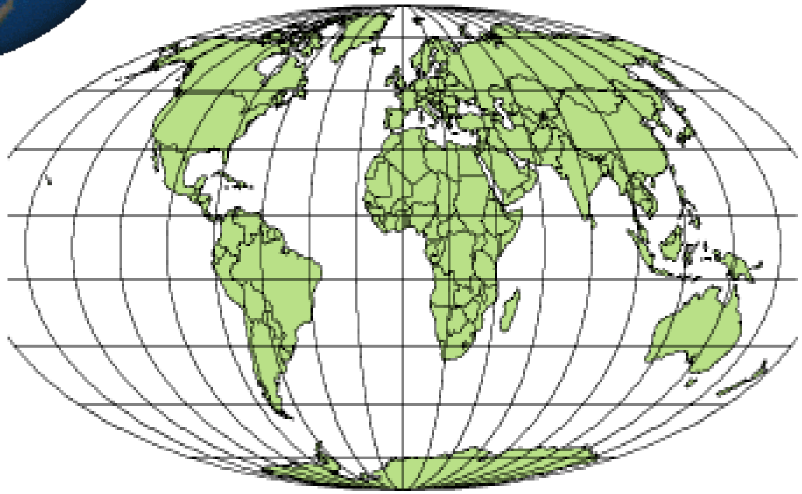
Parametrization

Paolo Cignoni

3D GEOMETRIC MODELING & PROCESSING

What is a parametrization?



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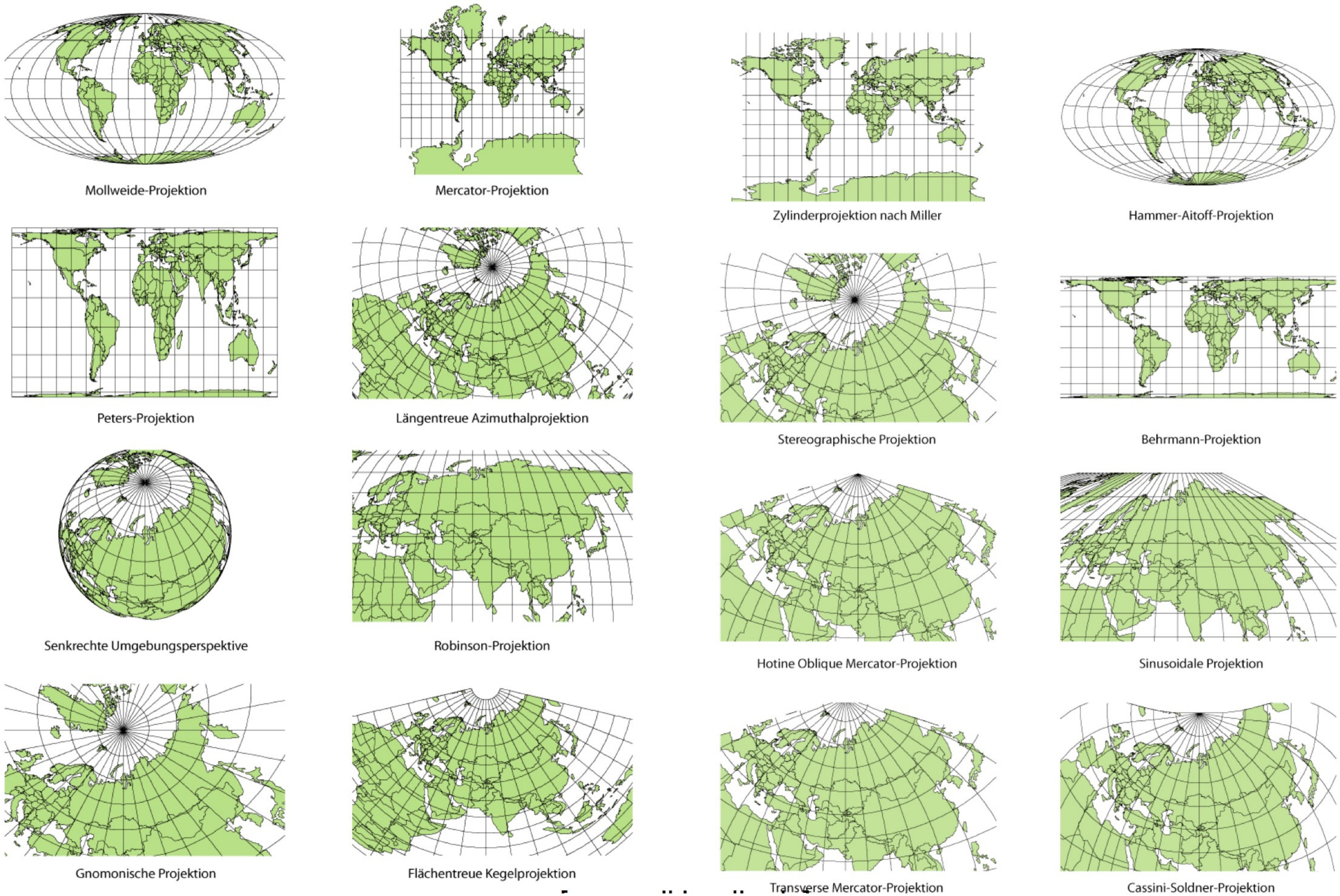
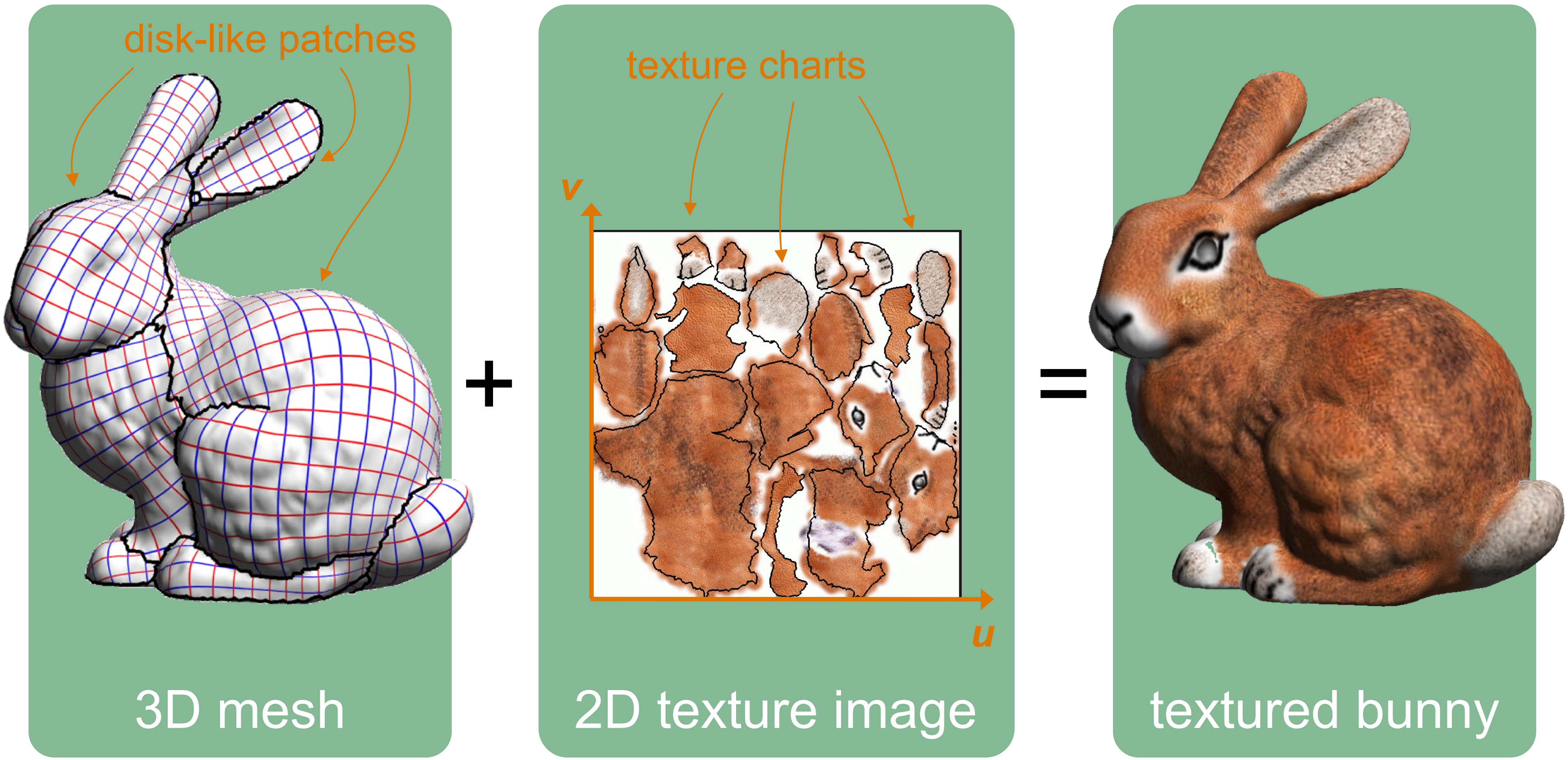


http://vcg.isti.cnr.it/~tarini/spinnableworldmaps/

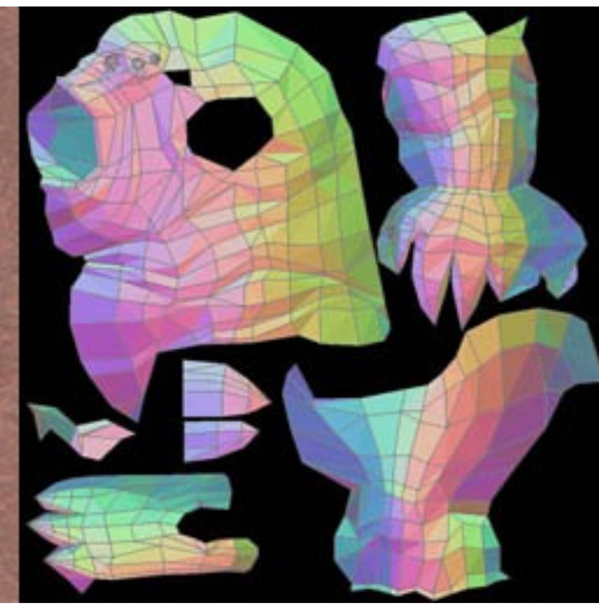
Why Parametrization?

- Texture Mapping



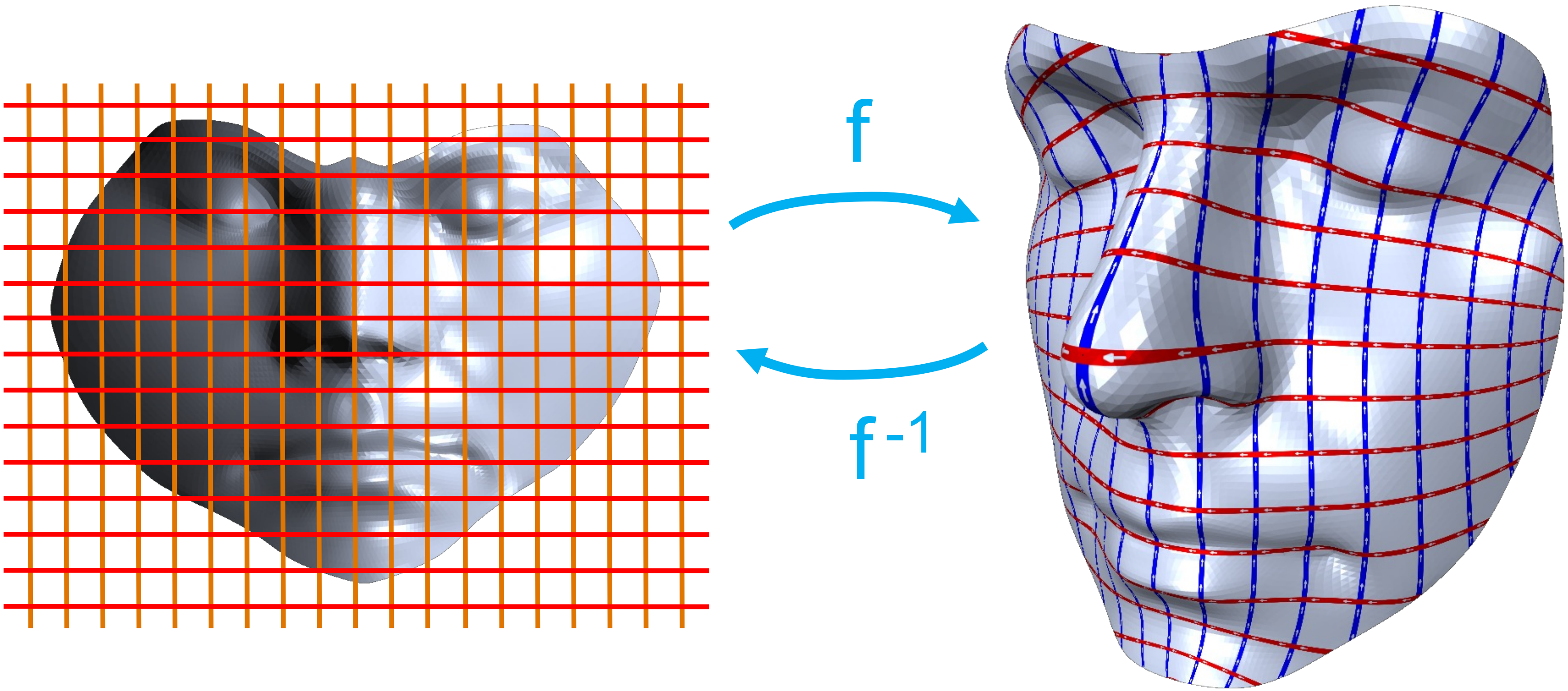
Why Parametrization?

- Manual UV mapping
- An advanced artistic skill

Why Parametrization?

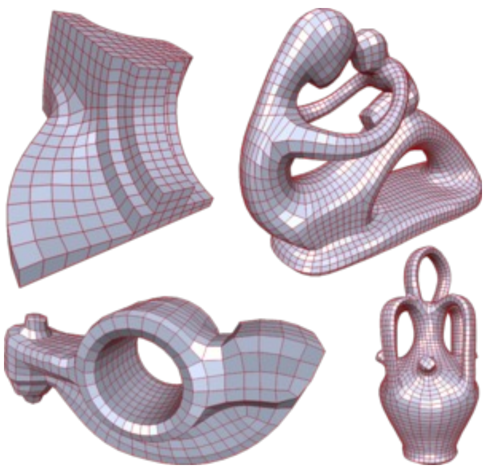
- Remeshing



Why Parametrization?

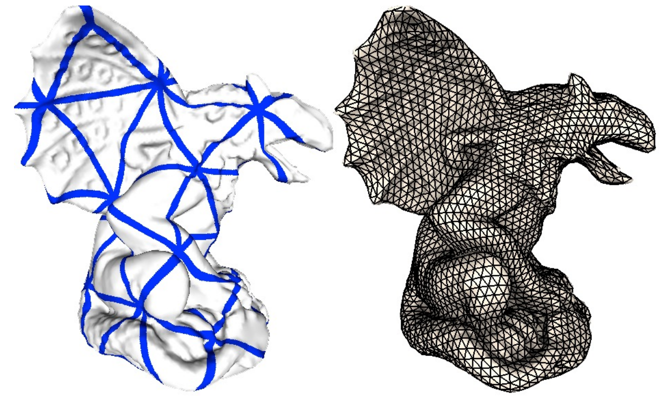
- Remeshing

QUADRILATERAL



Bommes, et AL.: *Mixed Integer Quadrangulation*

TRIANGULAR

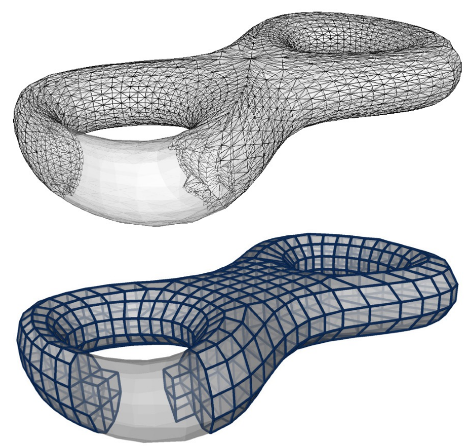


Pietorni, et AL. :Almost isometric mesh parameterization through abstract domains

HEXAGONAL

Nieser et al.: *Hexagonal Global Parameterization of Arbitrary Surfaces*

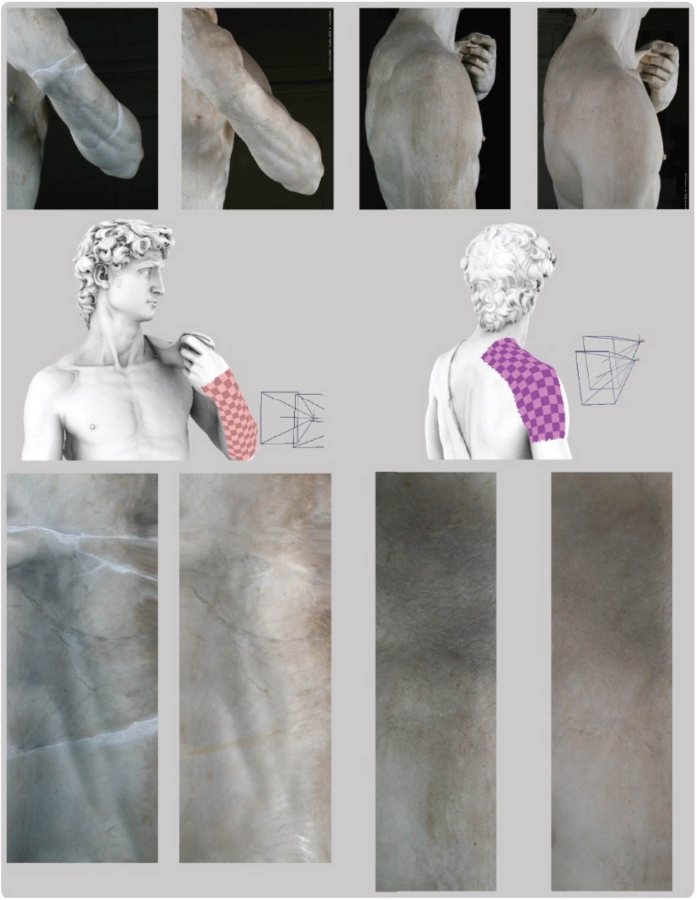
HEXAHEDRAL



Nieser, et AL. : CUBECOVER – Parameterization of 3D Volumes

Why Parametrization?

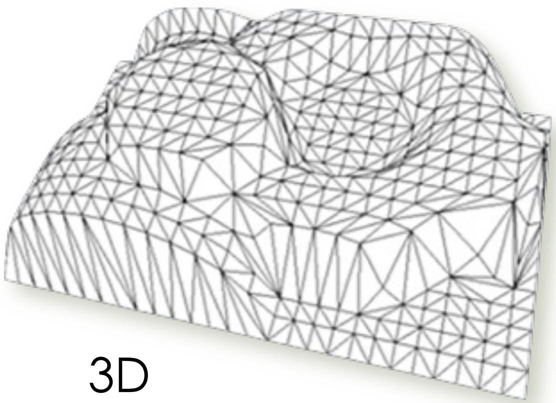
- Analysis…. 2D is easier than 3D



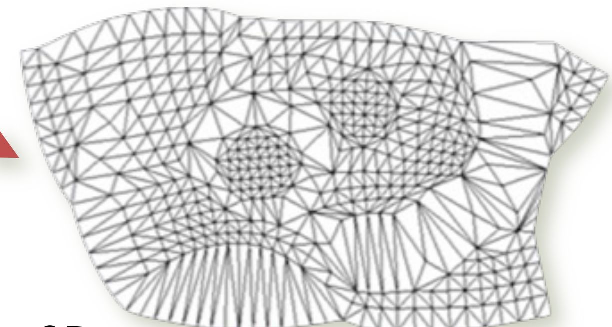
Pietroni, et AL.: *An Interactive Local Flattening Operator to Support Digital Investigations on Artwork Surfaces*

Parametrization: what we need?

- A strategy to flatten a 3D surface on 2D domain
  - Introducing as few distortion as possible

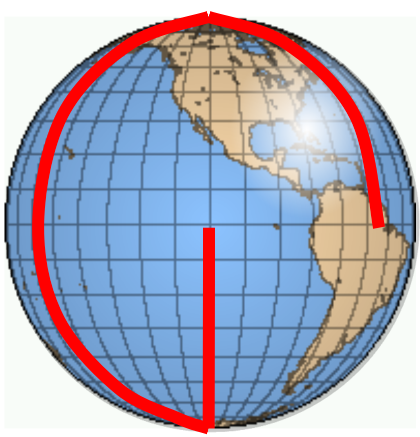


3D

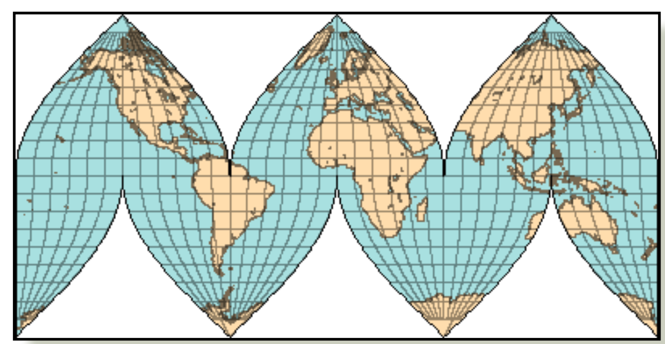


2D

- A strategy to introduce cuts



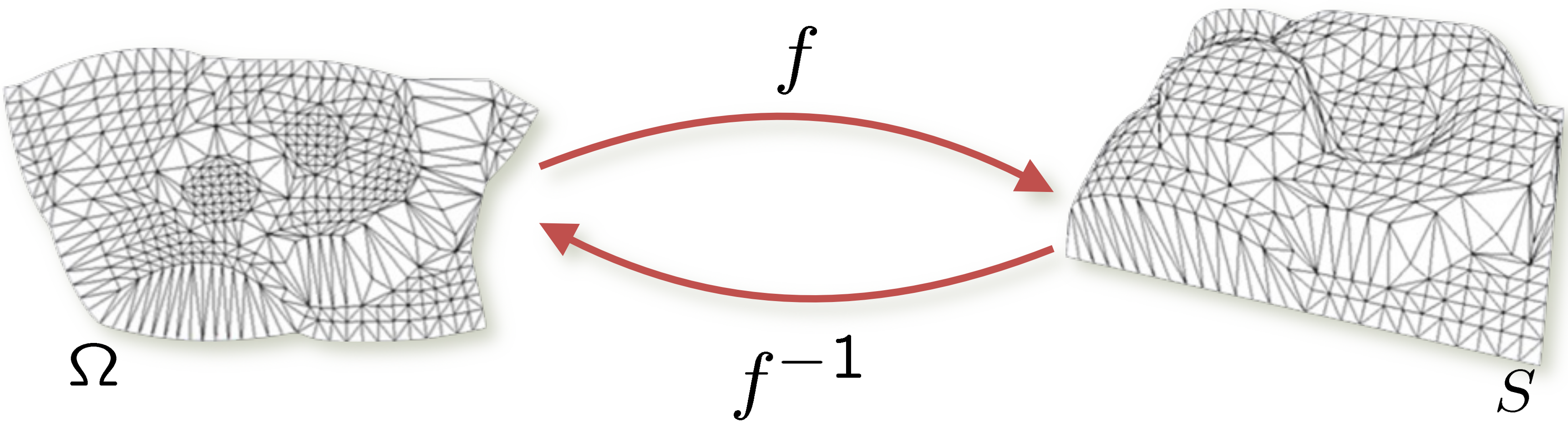
3D



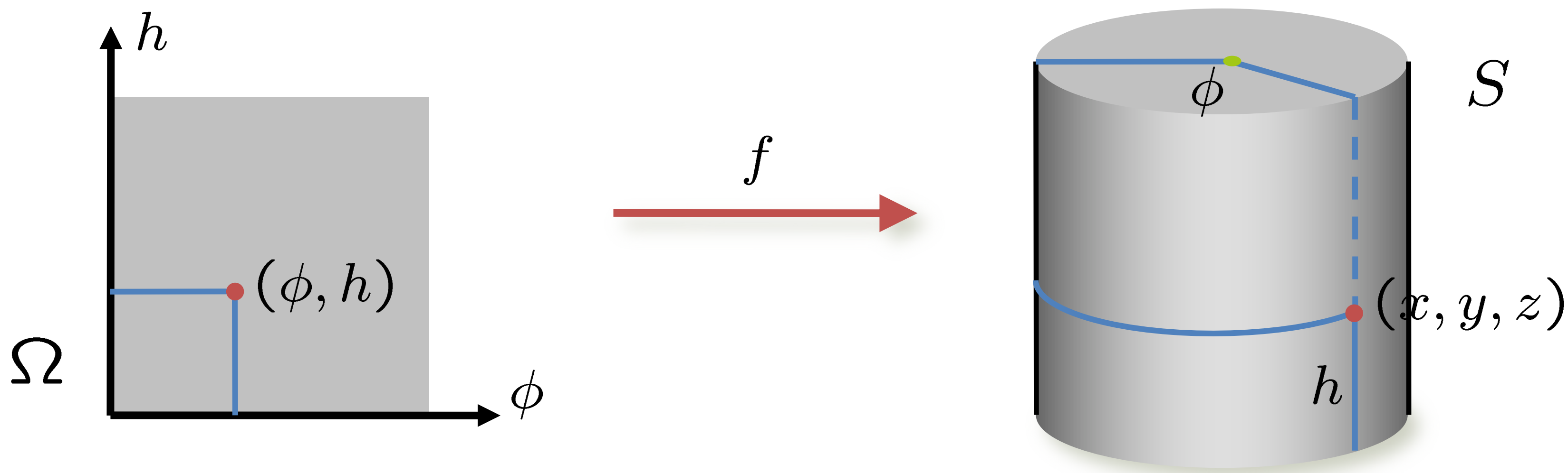
2D

Flattening a surface

- surface $S \subset \mathbb{R}^3$
- parameter domain $\Omega \subset \mathbb{R}^2$
- mapping $f : \Omega \to S$ and $f^{-1} : S \to \Omega$



Parametrization: Cylindrical coords



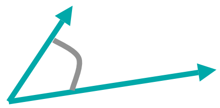
$$S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1, z \in [0, 1]\}$$

$$\Omega = \{(\phi, h) \in \mathbb{R}^2 : \phi \in [0, 2\pi), h \in [0, 1]\}$$

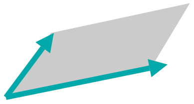
$$f(\phi, h) = (\sin\phi, \cos\phi, h)$$

Minimize Distortion

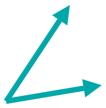
- Angle preservation: conformal



- Area preservation: equiareal



- Area and Angle: Isometric



Distortion

What happens to the surface point *f(u,v)* as we move a tiny little bit away from *(u,v)* in the parameter domain?

- Approximate with first order Taylor expansion

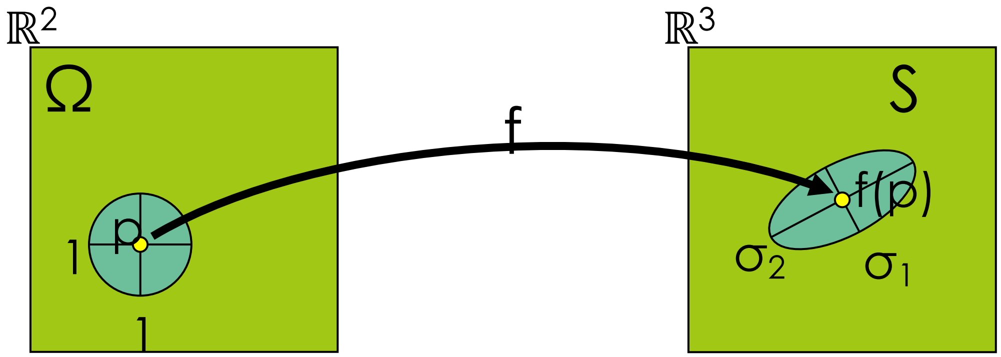
$$\tilde{f}(u+\Delta u, v+\Delta v) = f(u,v) + f_u(u,v)\Delta u + f_v(u,v)\Delta v.$$

$$f_u = \frac{\partial f}{\partial u} \quad \text{and} \quad f_v = \frac{\partial f}{\partial v}$$

$$\tilde{f}(u+\Delta u, v+\Delta v) = \boldsymbol{p} + J_f(\boldsymbol{u})\binom{\Delta u}{\Delta v},$$

$$J_f = U\Sigma V^T = U \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{pmatrix} V^T,$$

  - J_f Jacobian of f, i.e. the 3×2 matrix with partial derivatives of f as column vectors



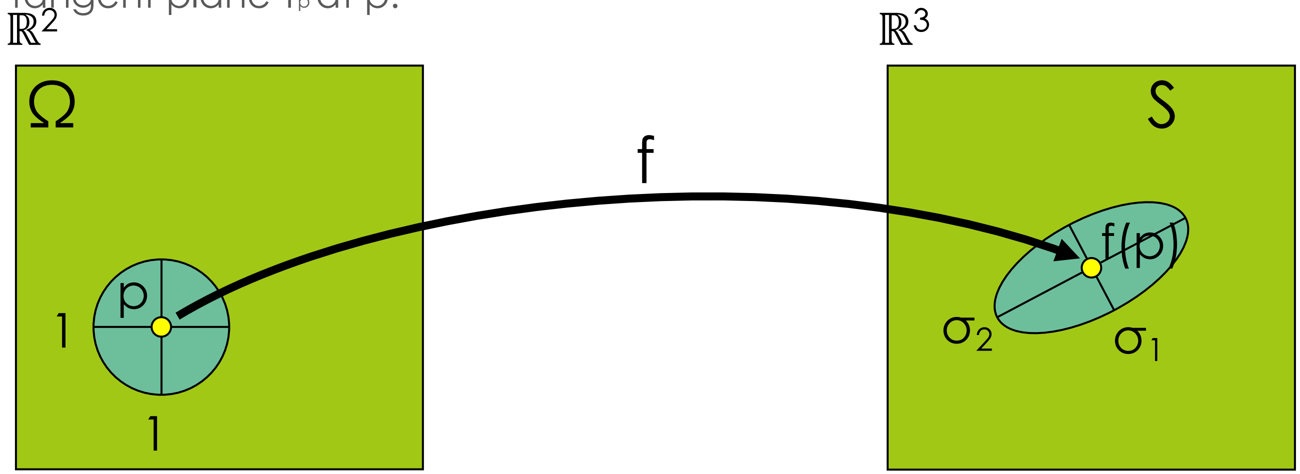
Distortion

$$\tilde{f}(u+\Delta u, v+\Delta v) = \boldsymbol{p} + J_f(\boldsymbol{u})\binom{\Delta u}{\Delta v}, \qquad J_f = U\Sigma V^T = U\begin{pmatrix}\sigma_1 & 0\\ 0 & \sigma_2\\ 0 & 0\end{pmatrix}V^T,$$

- Consider singular value decomposition of the Jacobian

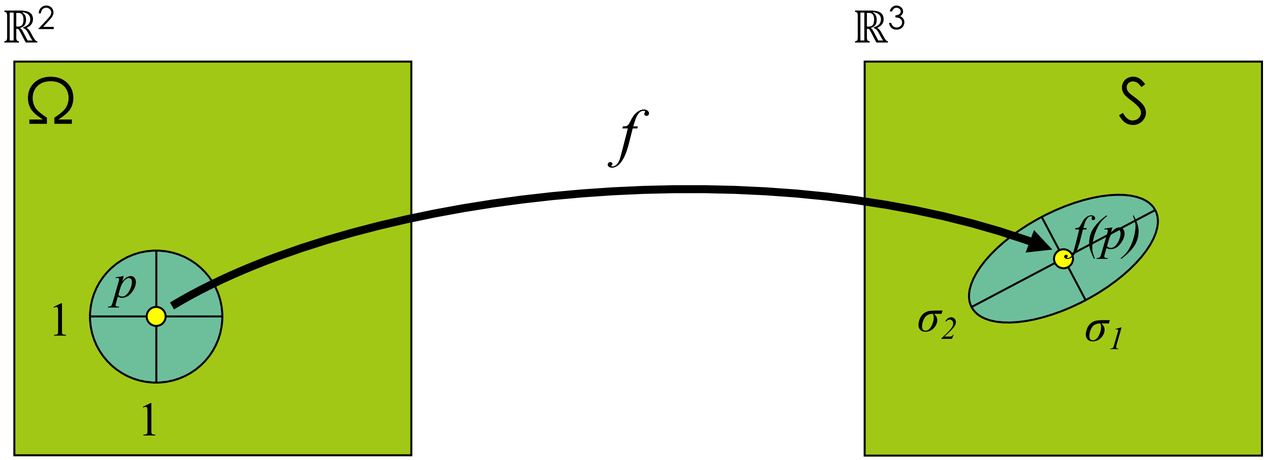
singular values $\sigma_1 \geq \sigma_2 > 0$ and *orthonormal* matrices $U \in \mathbb{R}^{3\times 3}$ and $V \in \mathbb{R}^{2\times 2}$

  - The transformation V^T first rotates all points around u such that the vectors V_1 and V_2 are in alignment with the u- and the v-axes afterwards.
  - The transformation Σ then **stretches** by the factor σ_1 in the u- and by σ_2 in the v-direction.
  - The transformation U finally maps the unit vectors (1, 0) and (0, 1) to the vectors U_1 and U_2 in the tangent plane T_p at p.



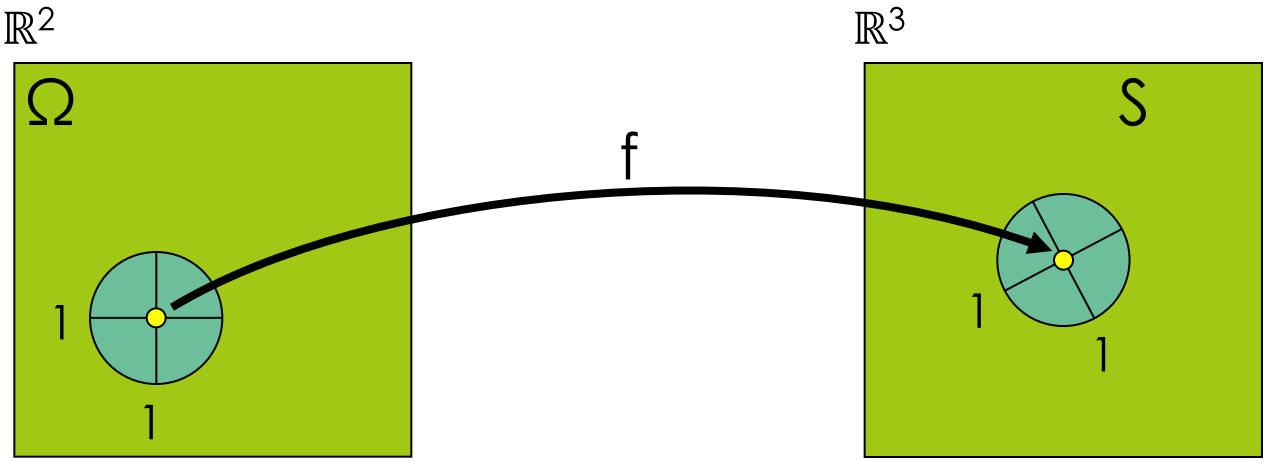
Distortion

- In practice the values σ_1 and σ_2 describe the amount of the local deformations



Isometric Mapping

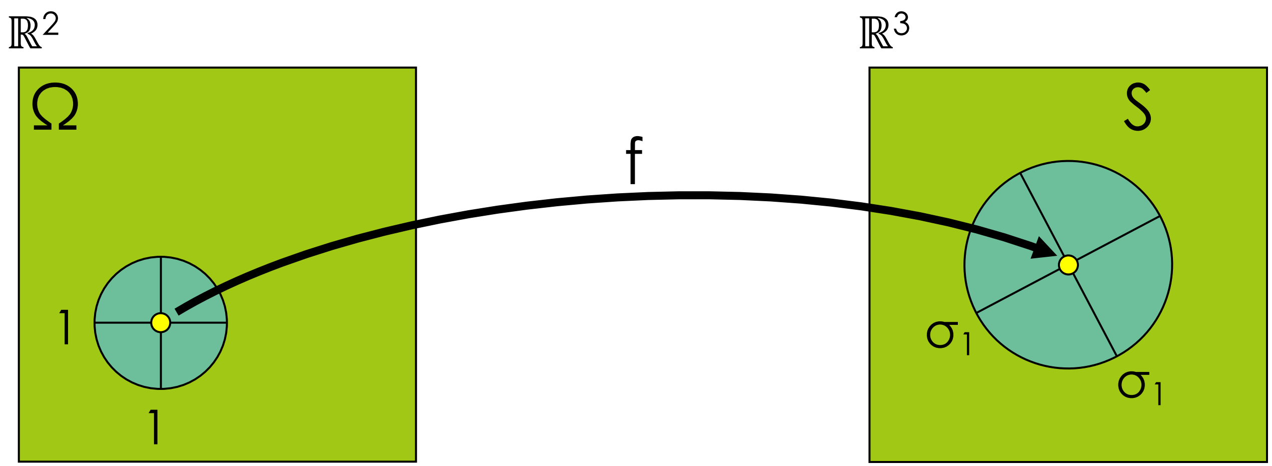
- $\sigma_1 = \sigma_2 = 1$



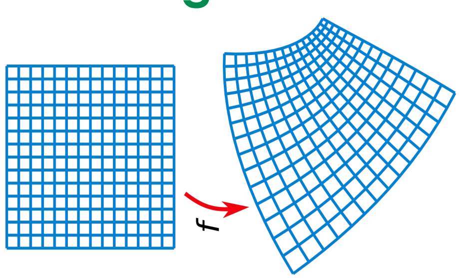
- preserves **areas**, **angles** and **lengths**

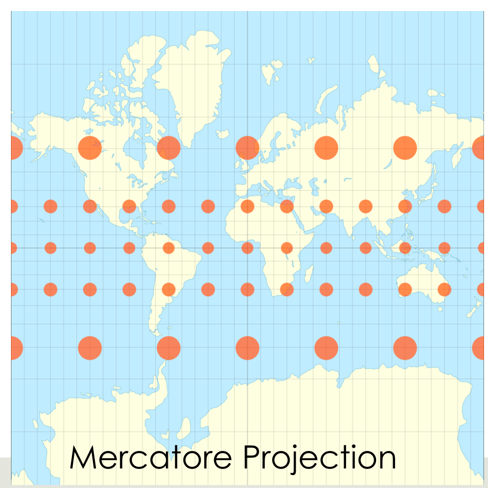
Conformal Mapping

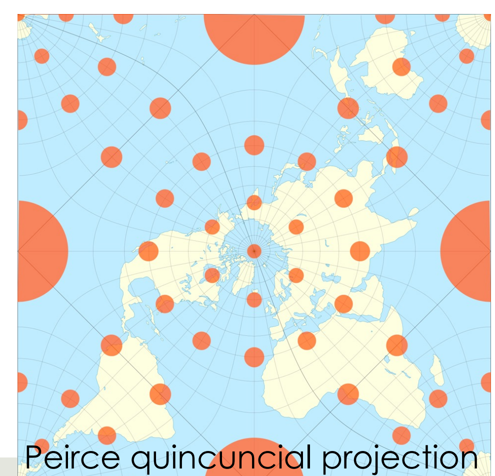
- $\sigma_1 / \sigma_2 = 1$



- preserves **angles**

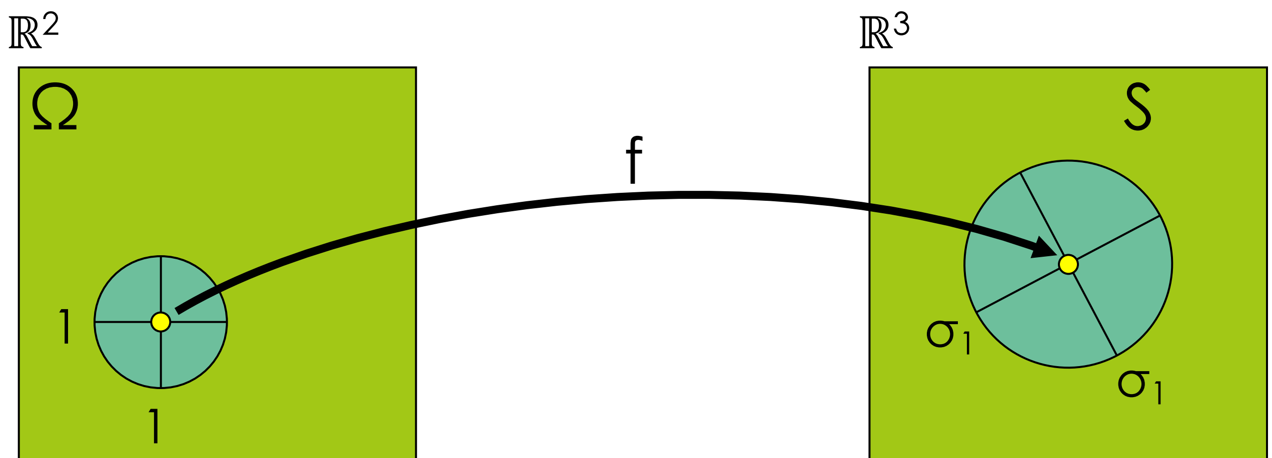



Mercatore Projection

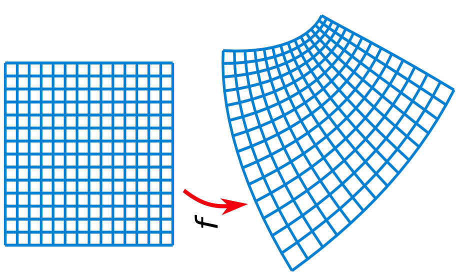

Peirce quincuncial projection

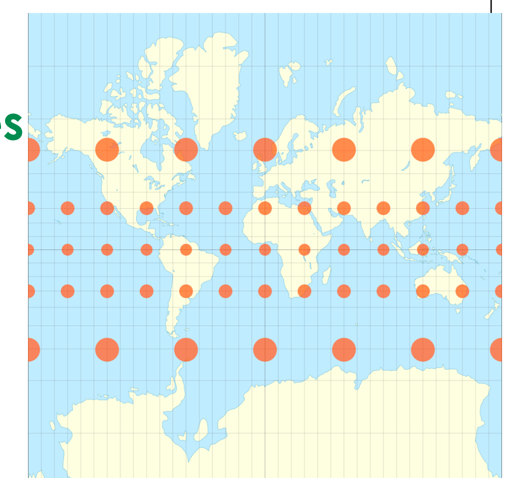
Conformal Mapping

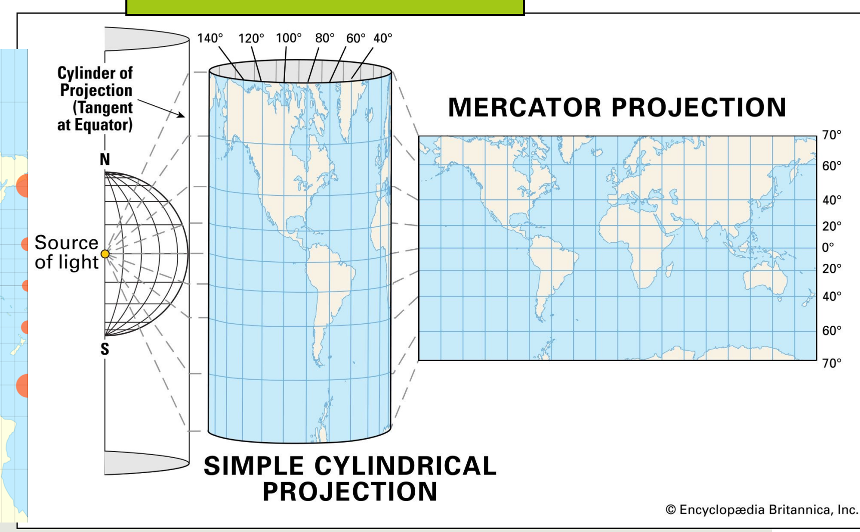
- $\sigma_1 / \sigma_2 = 1$



- preserves **angles**

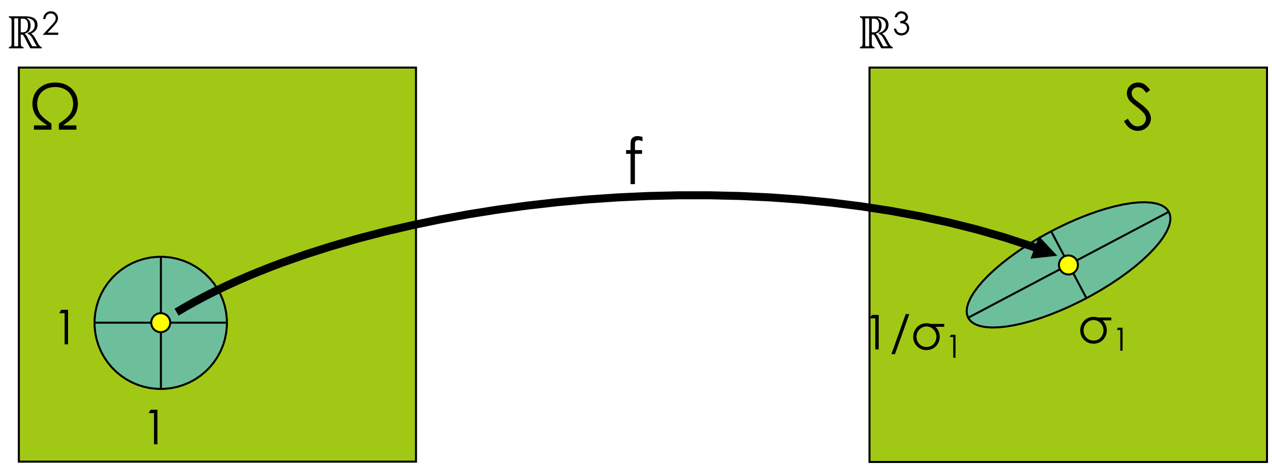




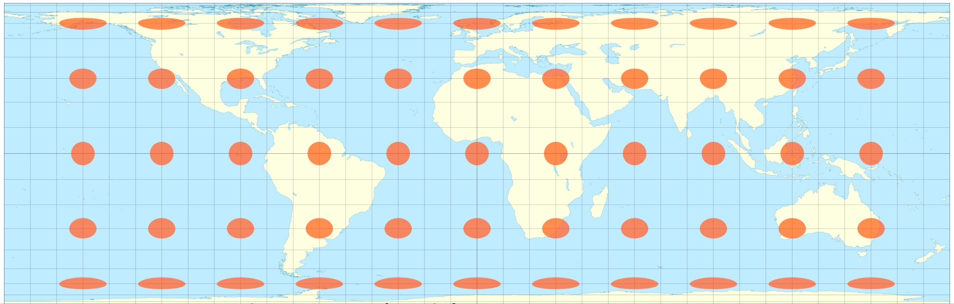


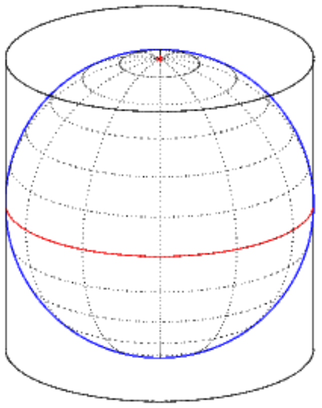
Equiareal Mapping

- $\sigma_1 \cdot \sigma_2 = 1$



- preserves **areas**

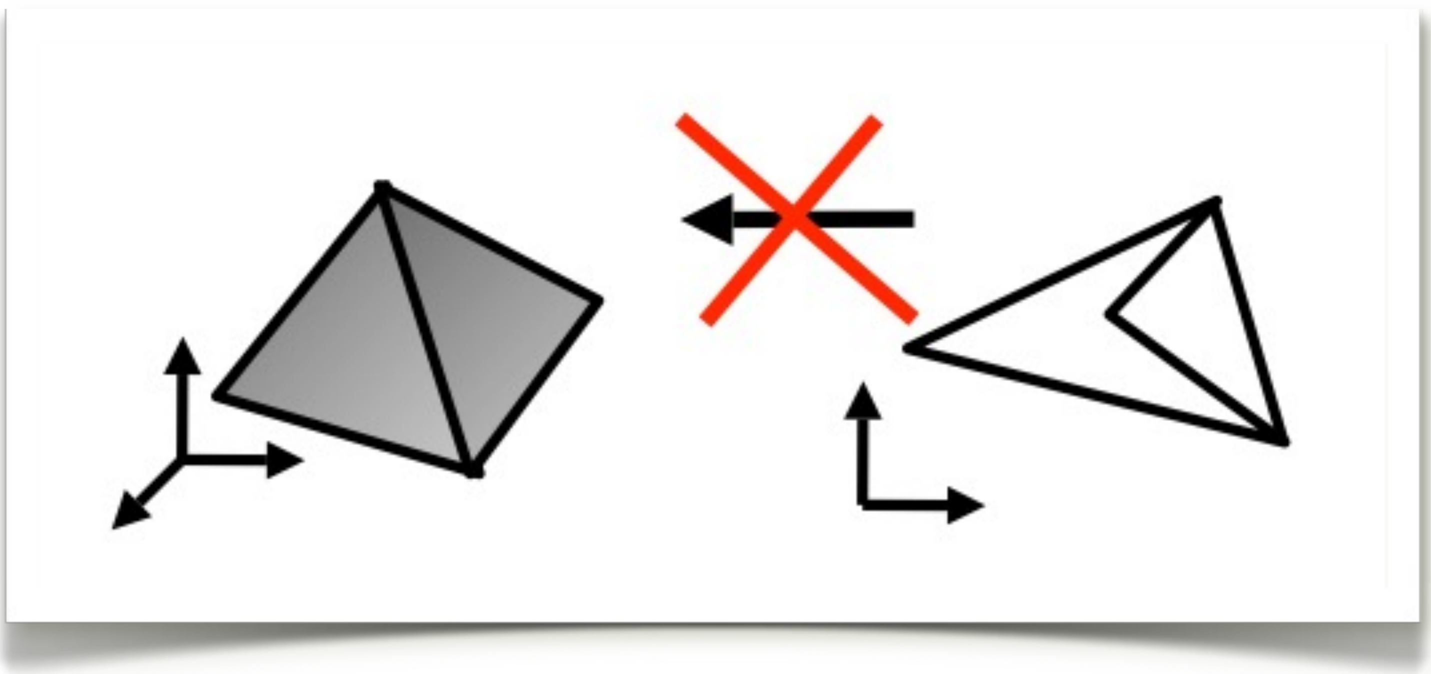




Lambert cylindrical equal-area

Bijectivity

- Parametrization map must be bijective ⇔ triangles in parametric domain do not overlap (no triangle flips)



Bijectivity

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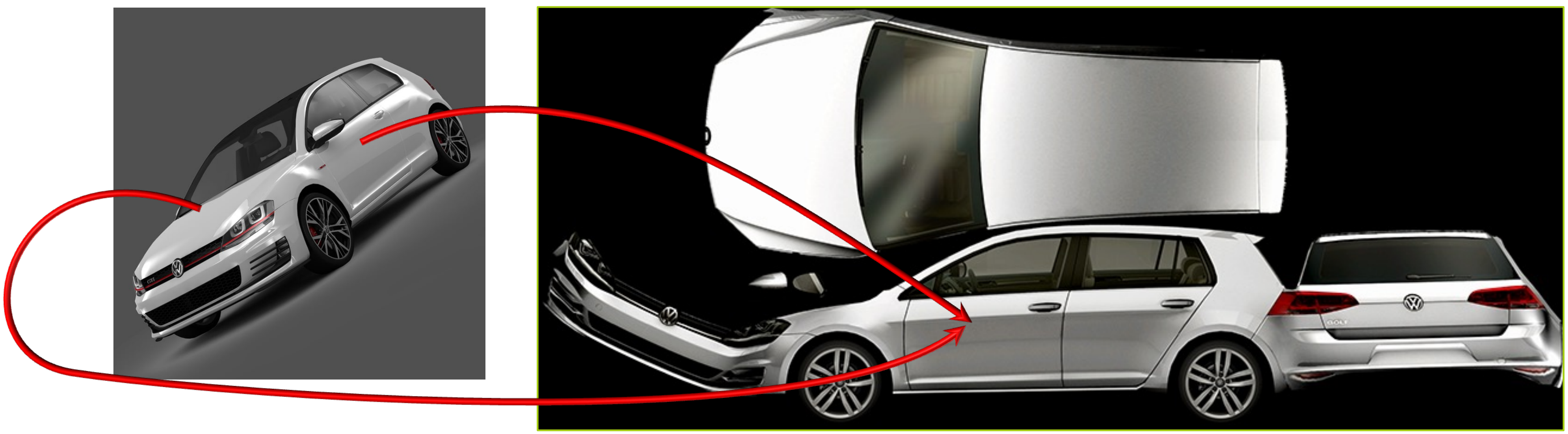
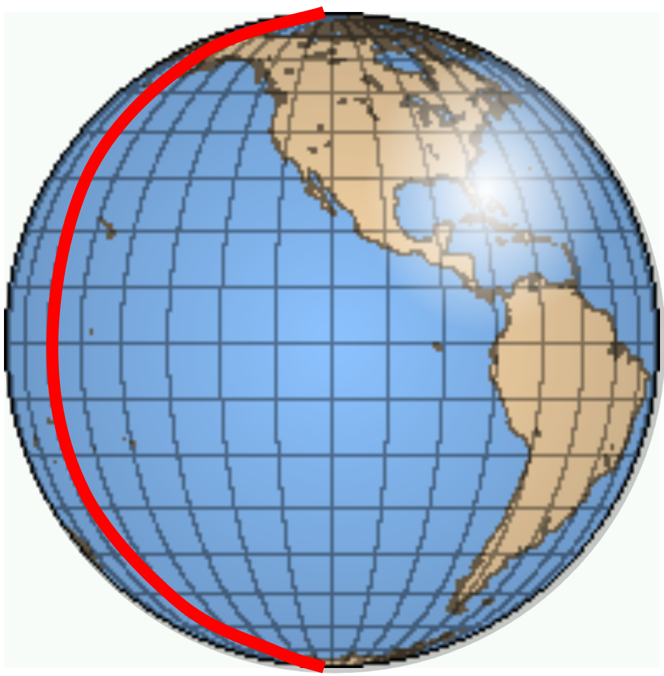
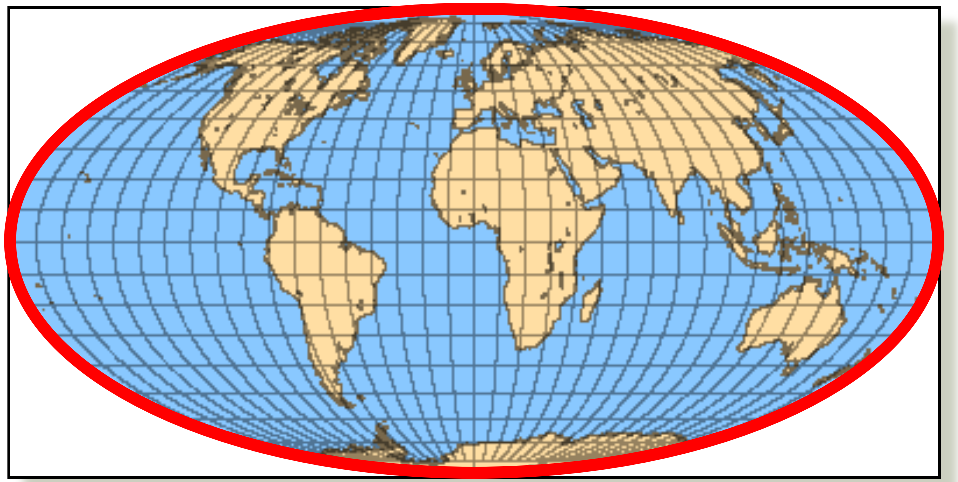


Cuts 1

- Clearly needed for closed surfaces



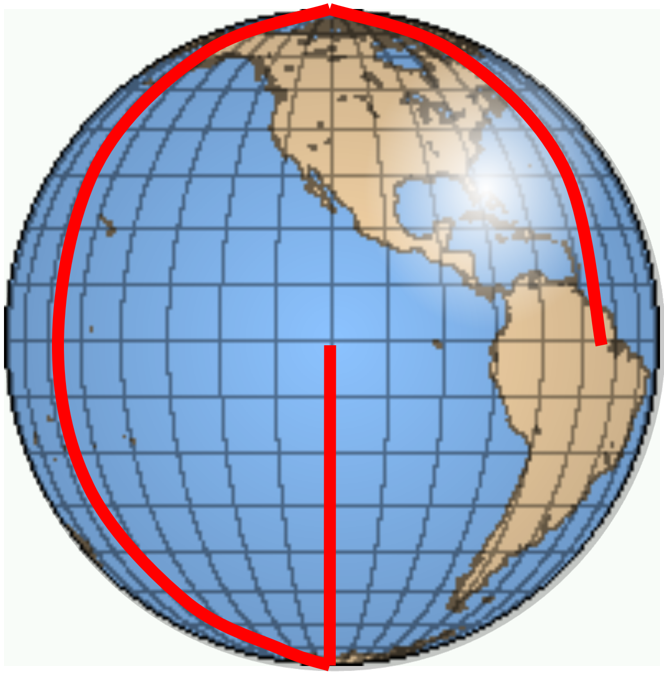
sphere in 3D



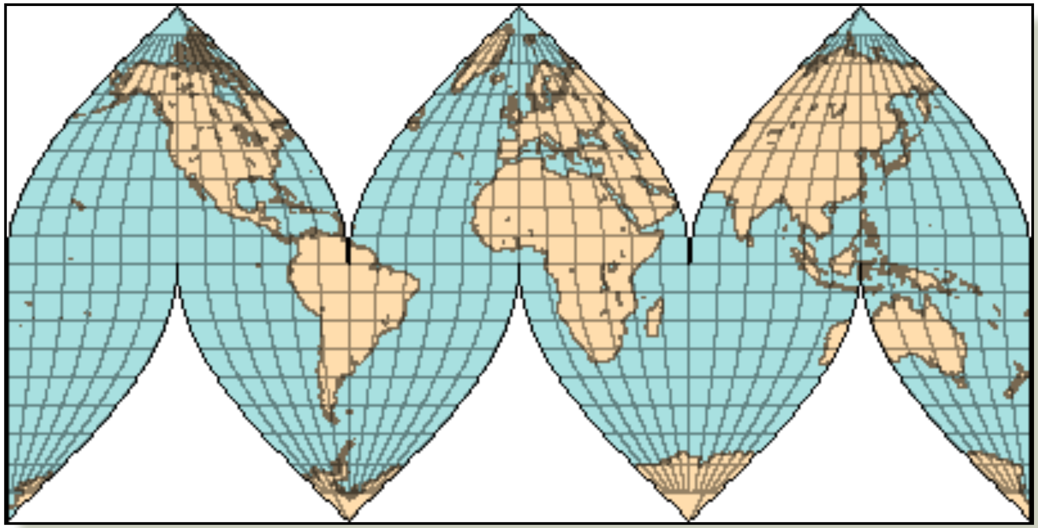
2D surface disk

Cuts 2

- Usually more cuts -> less distortion



sphere in 3D



2D surface

Cuts 3: closed surfaces

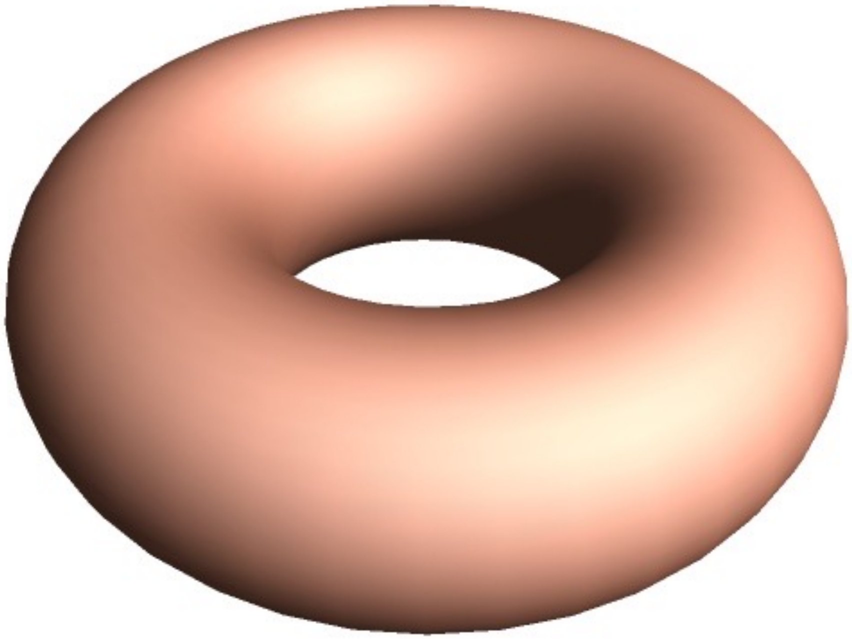
- How many cuts?

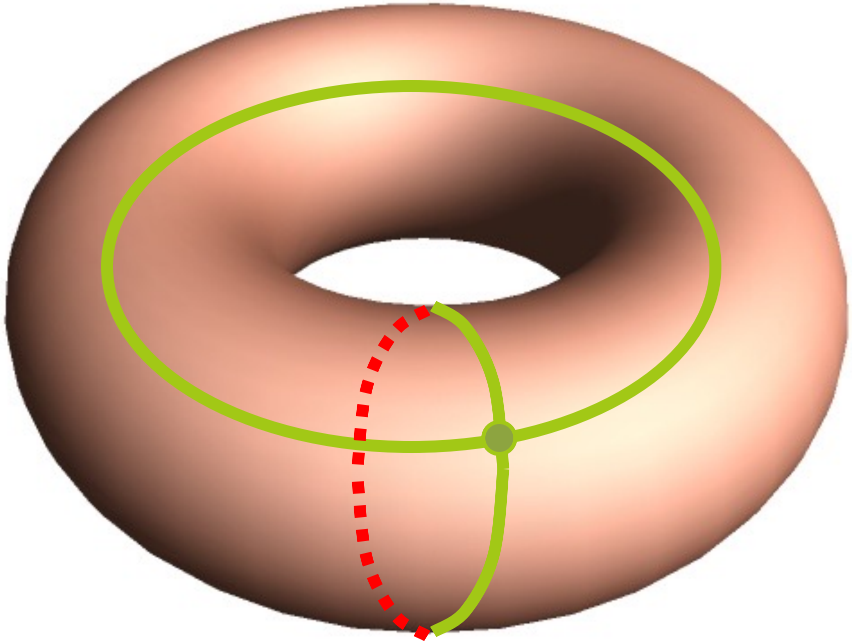
for a genus 0 surface ?

any tree of cuts
(more on this later)

Cuts 3: closed surfaces

- How many cuts?



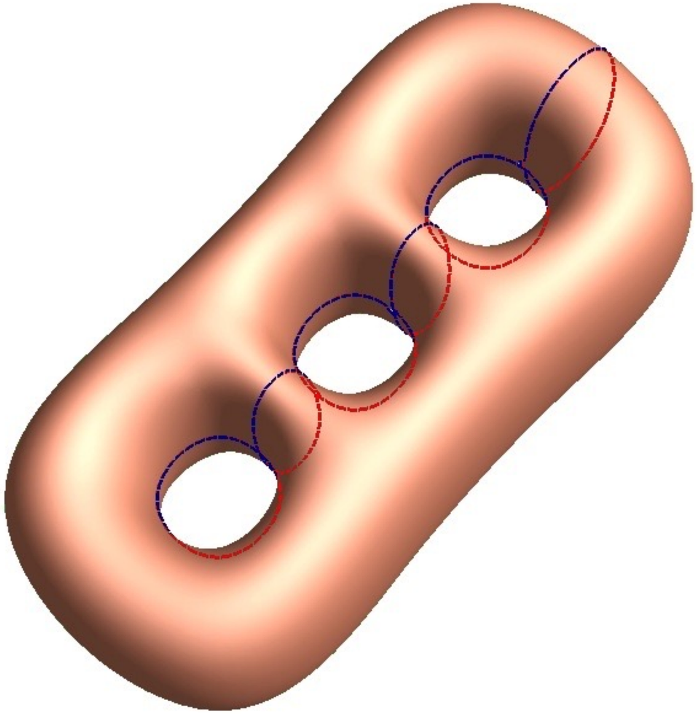


for a genus 1 surface ?

two looped cuts

Cuts 3: closed surfaces

- How many cuts?

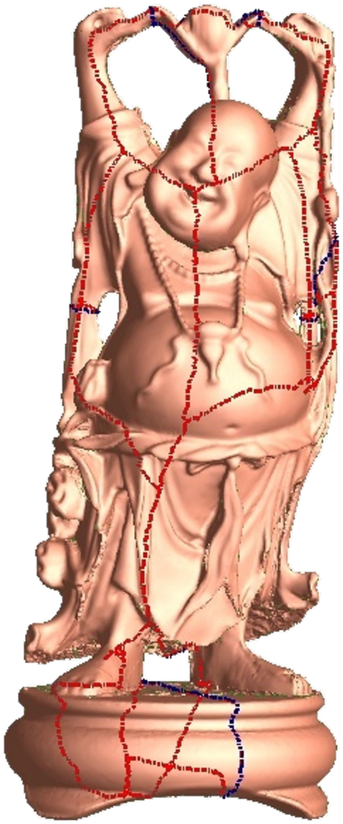


for a genus 3 surface ?

6 looped cuts

Cuts 3: closed surfaces

- How many cuts?



genus 6

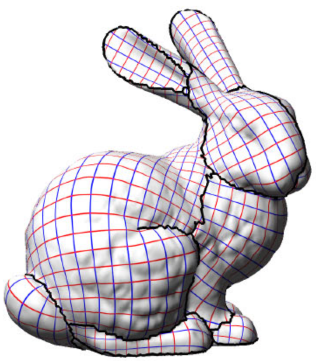
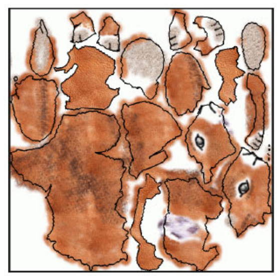
for a genus n surface ?

2n looped cuts

Generic Cut Strategies

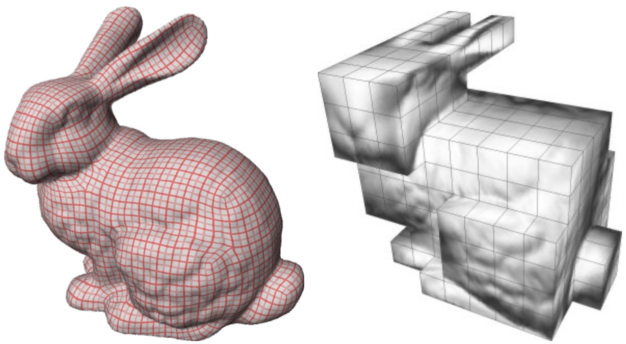
- Texture Mapping

UNSTRUCTURED CUTS

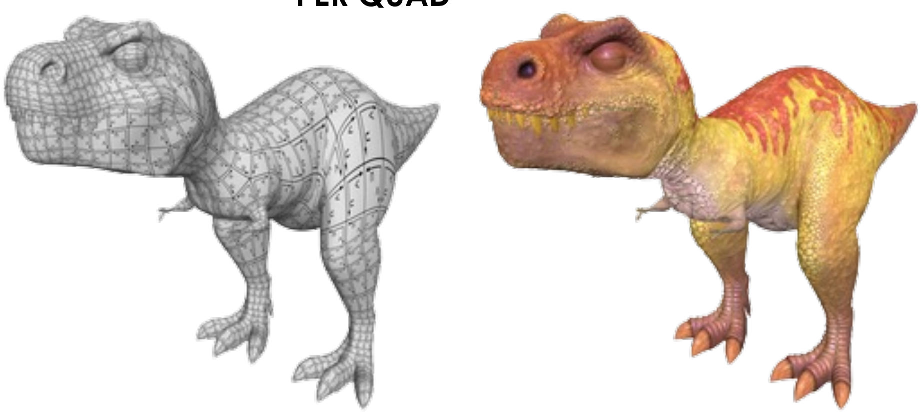

Lévy, et AL.: *Least squares conformal maps for automatic texture atlas generation*

IMPLICIT



Tarini, et AL.: *PolyCube Maps*

PER QUAD



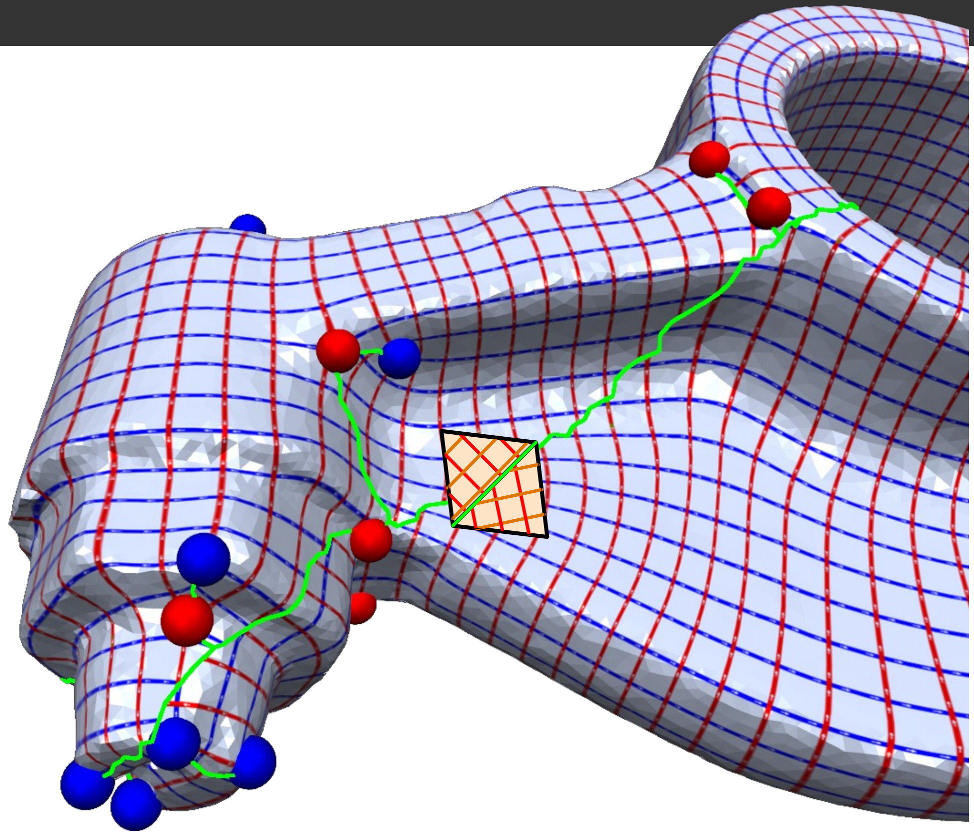
Brent Burley et al : Ptex: Per-Face Texture Mapping for Production Rendering

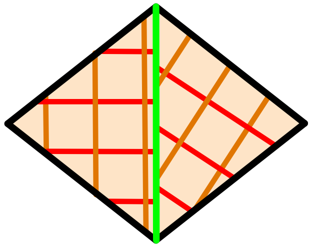
REGULAR CUTS


Pietroni, et AL.: *Almost isometric mesh parameterization through abstract domains*

Globally Smoothess

- Tangent directions varyes smoothly across seams





Globally Smoothess

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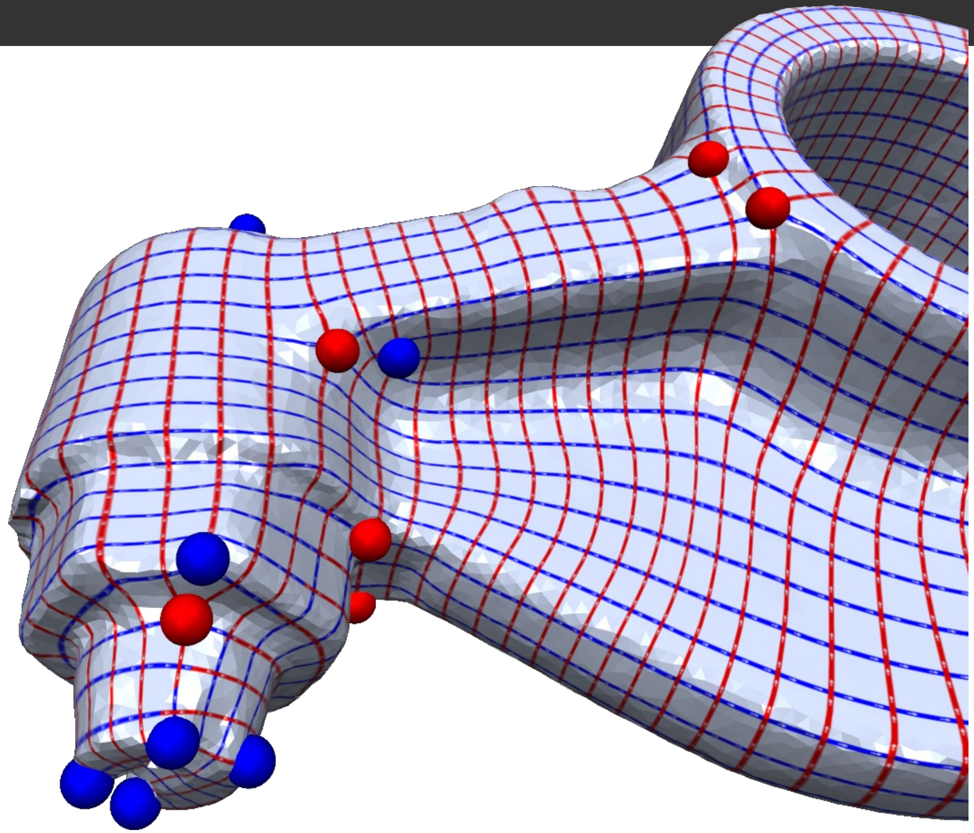
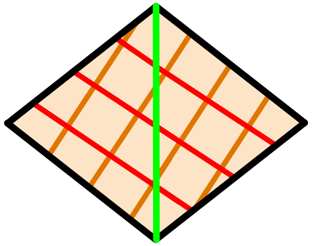
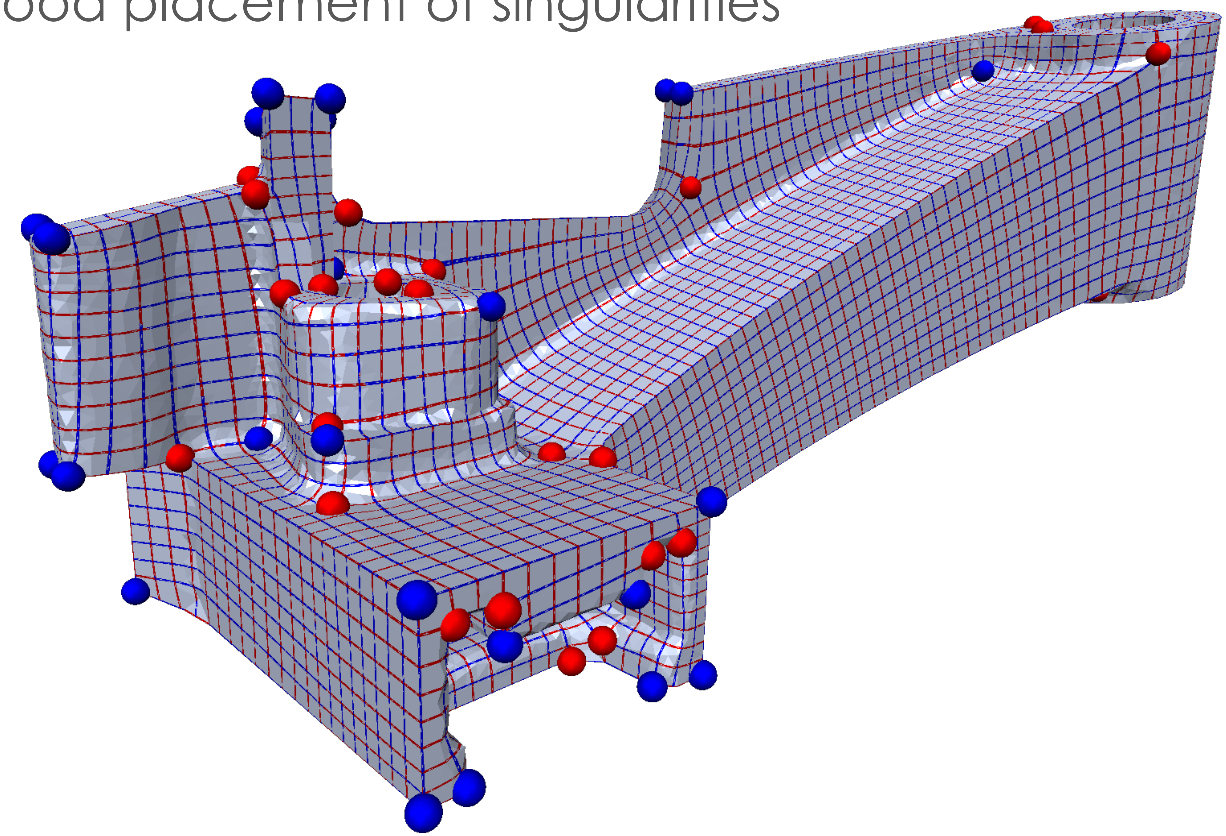




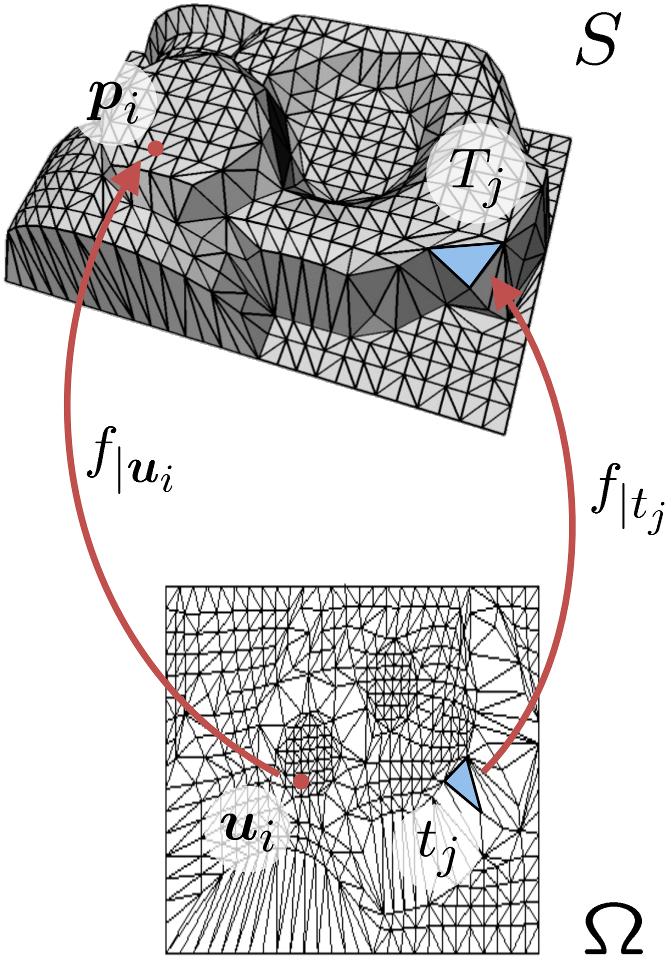
Feature Alignment

- Useful for quadrangulation
- Need good placement of singularities



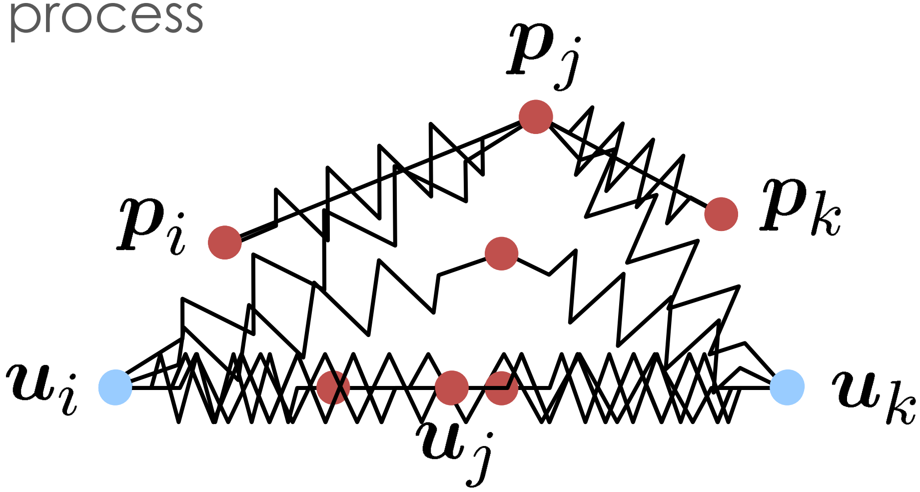
Details: Parametrization

- triangle mesh $S \subset \mathbb{R}^3$
  - vertices $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_{n+b}$
  - Triangles $T_1, \ldots, T_m$
- parameter mesh $\Omega \subset \mathbb{R}^2$
  - parameter points $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_{n+b}$
  - parameter triangles $t_1, \ldots, t_m$
- parameterization $f : \Omega \to S$
  - piecewise linear map $f(t_j) = T_j$



Parametrization: Mass-Spring

- replace **edges** by **springs**
- Position of vertices $p_0..p_n$
- UV Position of vertices $u_0..u_n$
- **relaxation** process



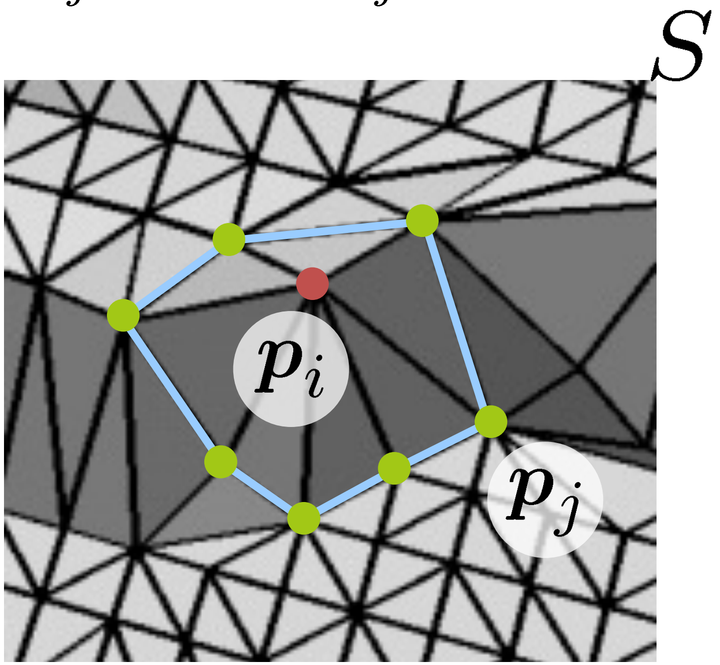
Energy Minimization

- energy of spring between p_i and p_j : $\frac{1}{2} D_{ij} {s_{ij}}^2$
- spring constant (stiffness) $D_{ij} > 0$
- spring length (in parametric space) $s_{ij} = \|\boldsymbol{u}_i - \boldsymbol{u}_j\|$
- total energy

$$E = \sum_{(i,j) \in \mathcal{E}} \frac{1}{2} D_{ij} \|\boldsymbol{u}_i - \boldsymbol{u}_j\|^2$$

- partial derivative

$$\frac{\partial E}{\partial \boldsymbol{u}_i} = \sum_{j \in N_i} D_{ij} (\boldsymbol{u}_i - \boldsymbol{u}_j)$$



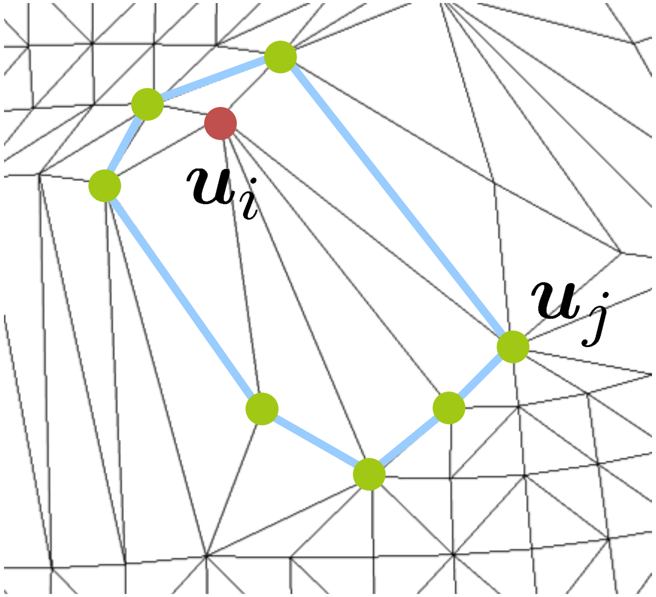
Linear System

- u_i is expressed as a **convex combination** of its neighbours u_j

$$\boldsymbol{u}_i = \sum_{j \in N_i} \lambda_{ij} \boldsymbol{u}_j$$

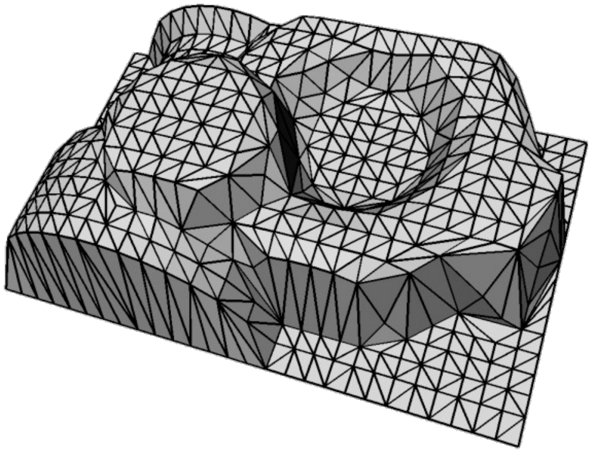
- With weights

$$\lambda_{ij} = D_{ij} \Big/ \sum_{k \in N_i} D_{ik}$$

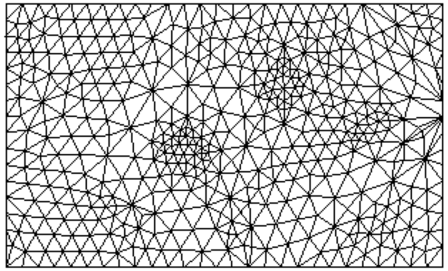


- LEAD to Linear System!

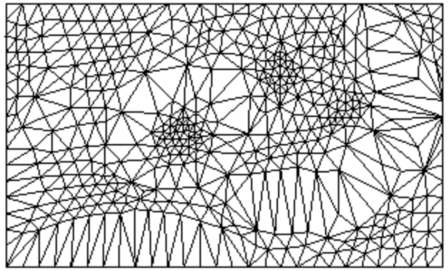
Which Weights?



- uniform spring constants



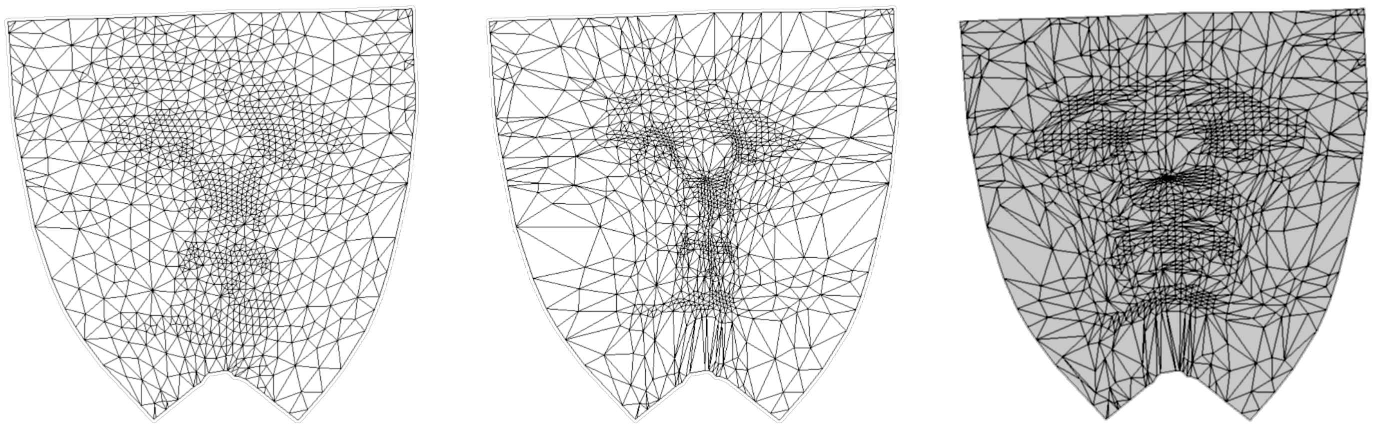
- Proportional to 3D distance



Which Weights?

- **NO** linear reproduction

- Planar mesh are distorted



Which Weights?

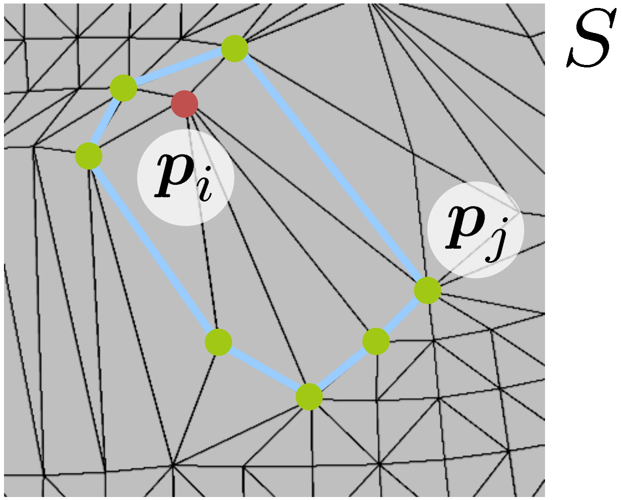
- suppose **S** to be is planar
- specify weights λ_{ij} such that

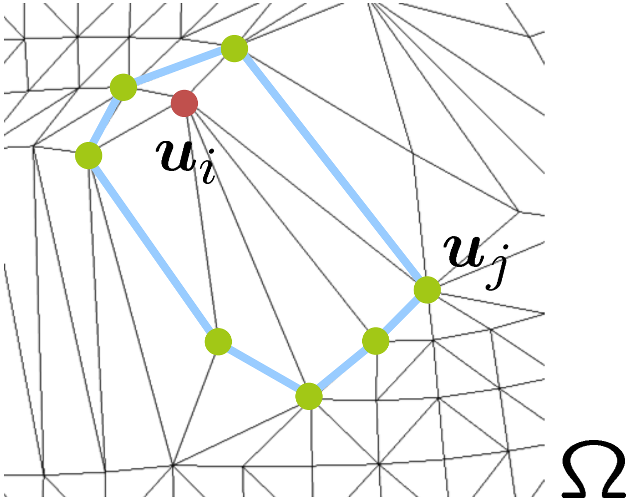
$$\boldsymbol{p}_i = \sum_{j \in N_i} \lambda_{ij} \boldsymbol{p}_j$$

- Then solving

$$\boldsymbol{u}_i = \sum_{j \in N_i} \lambda_{ij} \boldsymbol{u}_j$$

- Reproduces **S**





Which Weights?

- Wachspress coordinates

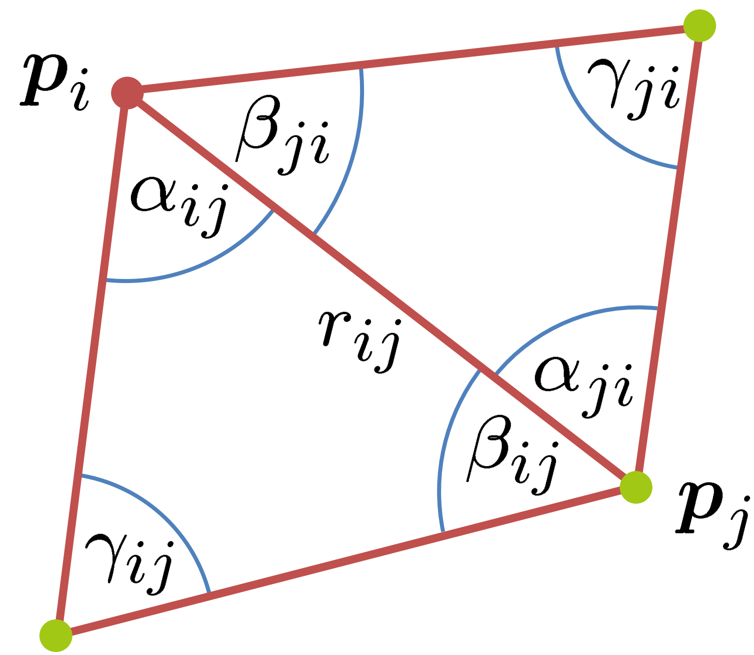
$$w_{ij} = \frac{\cot \alpha_{ji} + \cot \beta_{ij}}{{r_{ij}}^2}$$

- discrete harmonic coordinates

$$w_{ij} = \cot \gamma_{ij} + \cot \gamma_{ji}$$

- mean value coordinates

$$w_{ij} = \frac{\tan \frac{\alpha_{ij}}{2} + \tan \frac{\beta_{ji}}{2}}{r_{ij}}$$

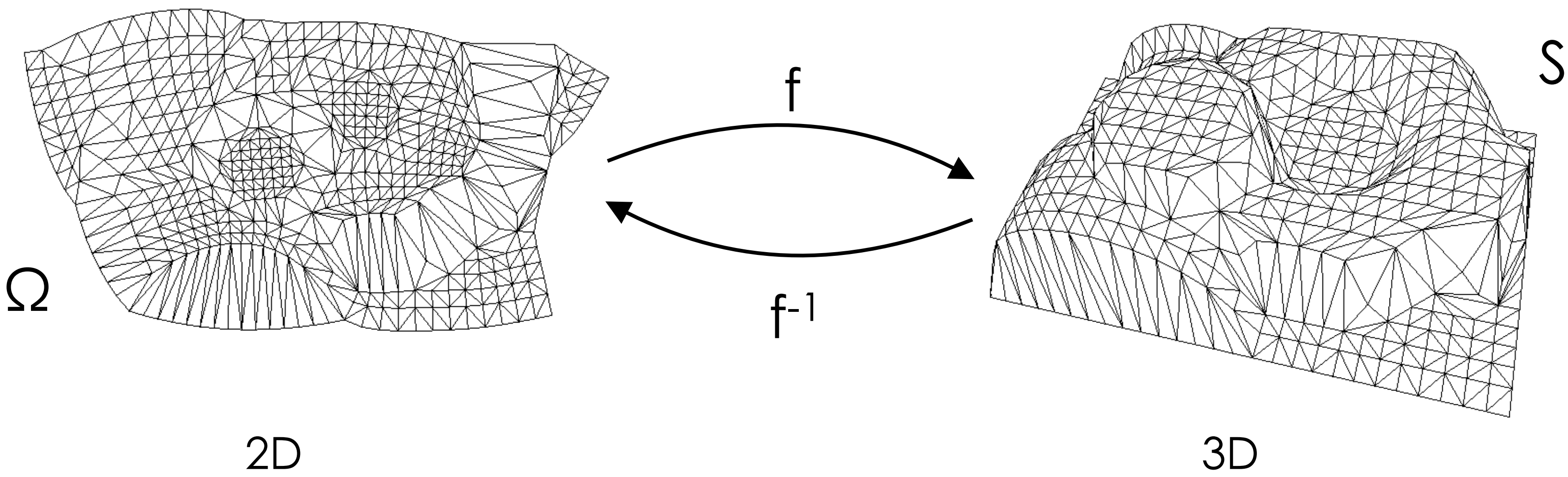


normalization

$$\lambda_{ij} = \frac{w_{ij}}{\sum_{k \in N_i} w_{ik}}$$

Recap

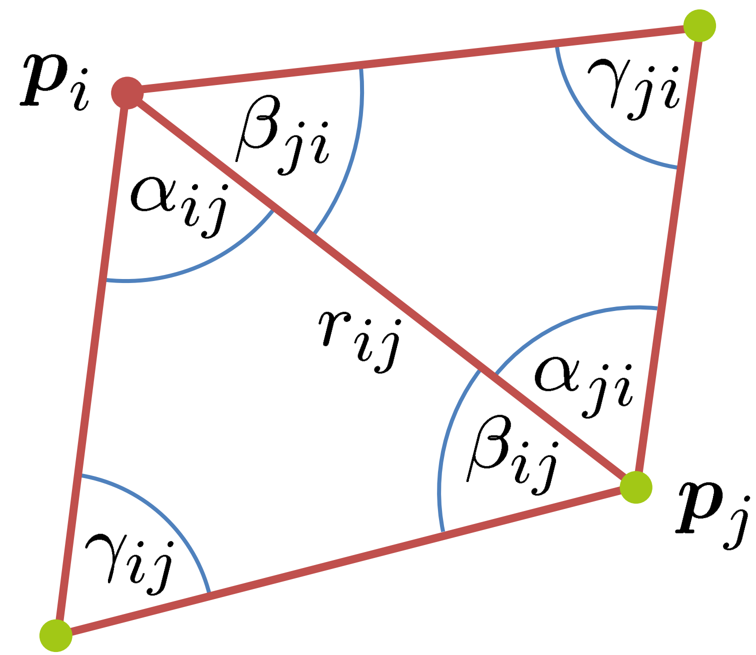
- Parametrization



Weighted average

- discrete harmonic coordinates

$$w_{ij} = \cot \gamma_{ij} + \cot \gamma_{ji}$$



normalization

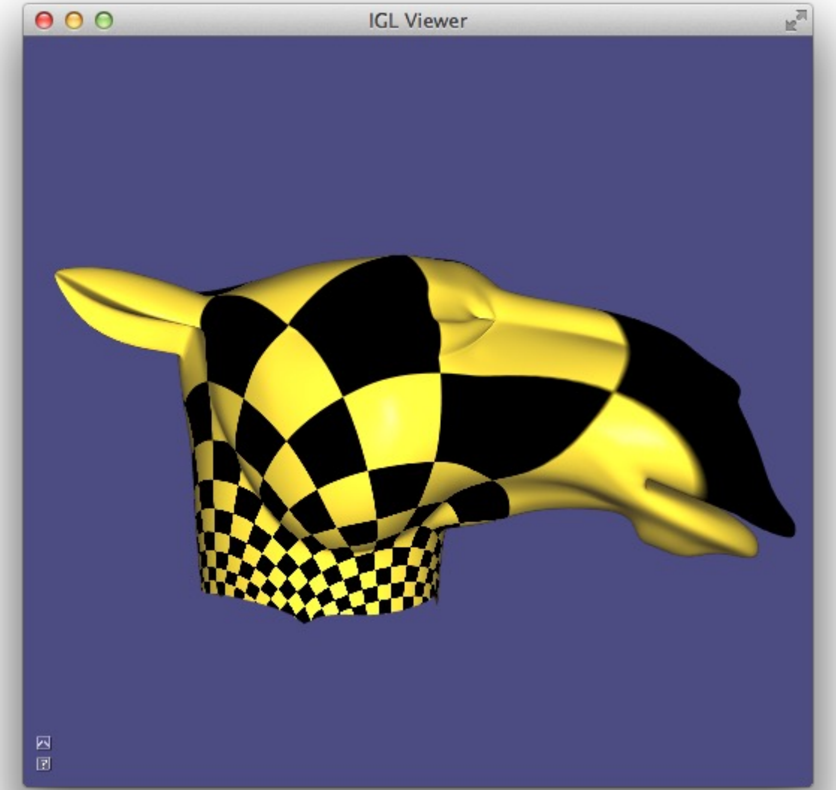
$$\lambda_{ij} = \frac{w_{ij}}{\sum_{k \in N_i} w_{ik}}$$

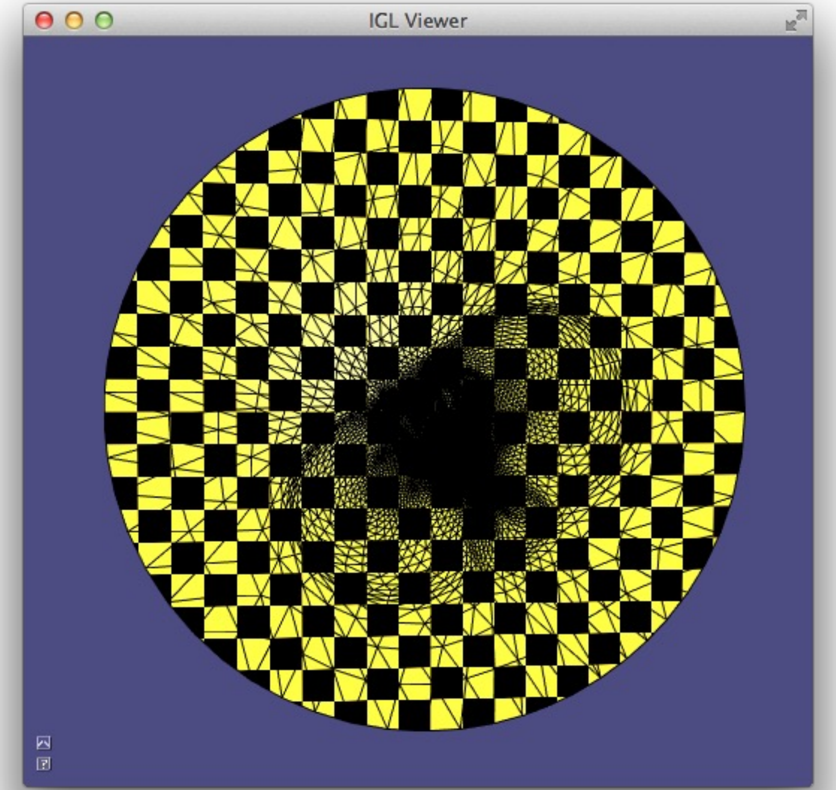
Harmonic parametrization

- Linear sistem
- Sparse matrix (2n x 2n), where n is number of vertices of the mesh
- Express each point as weighted sum of its neighbors
- Find x such that Ax=0
- x are the final UV coordinates!

Harmonic parametrization

- Fix the boundary of the mesh to UV
- Express each UV position as linear combination of neighbors
- Use cotangent weights!

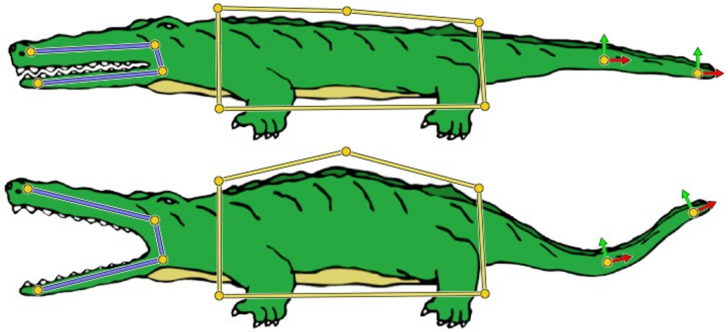




Harmonic Weights

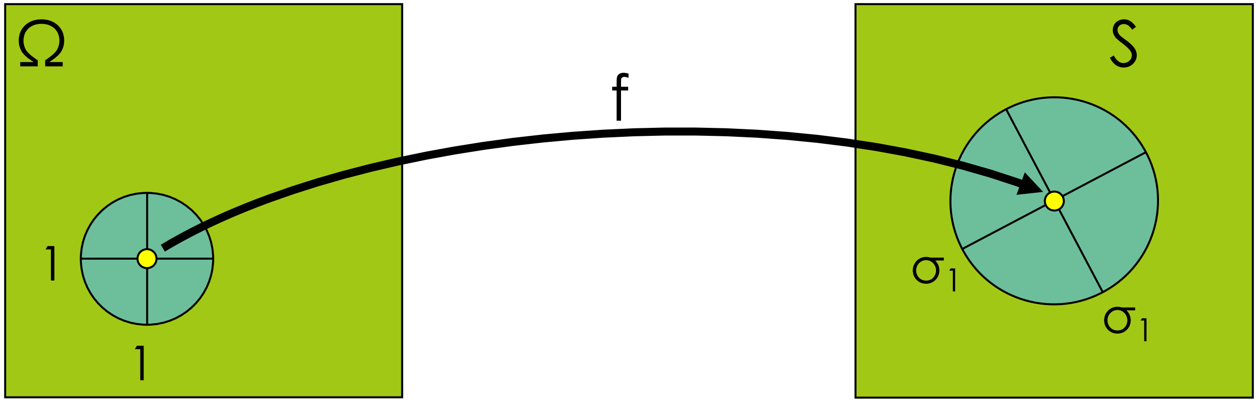
- Used to smoothly interpolate scalar values over a mesh given some sparse constraint

- Useful to interpolate deformations



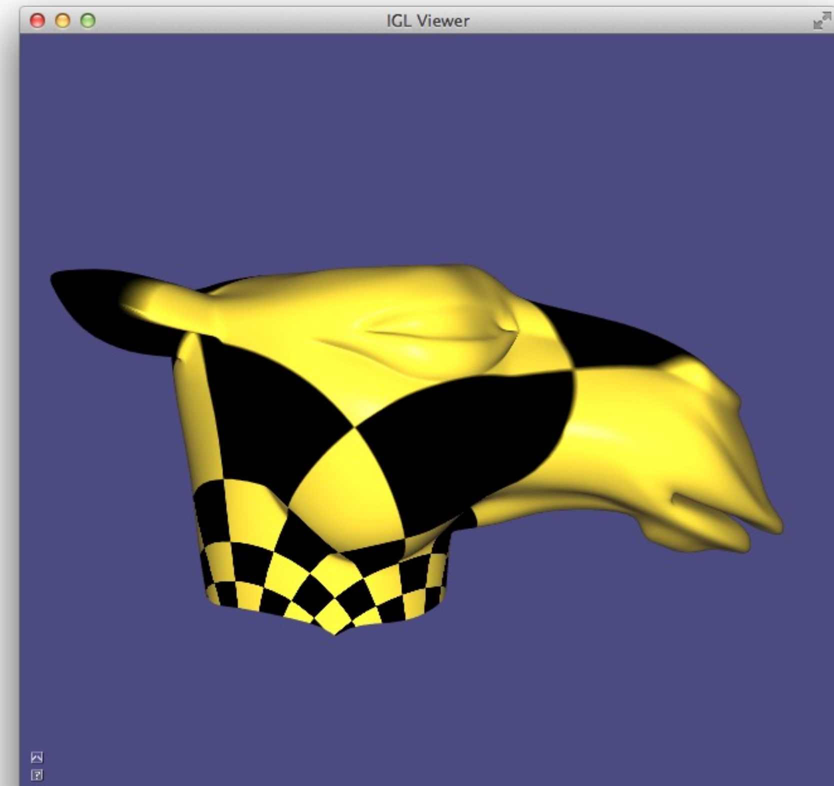
Least Squares Conformal maps

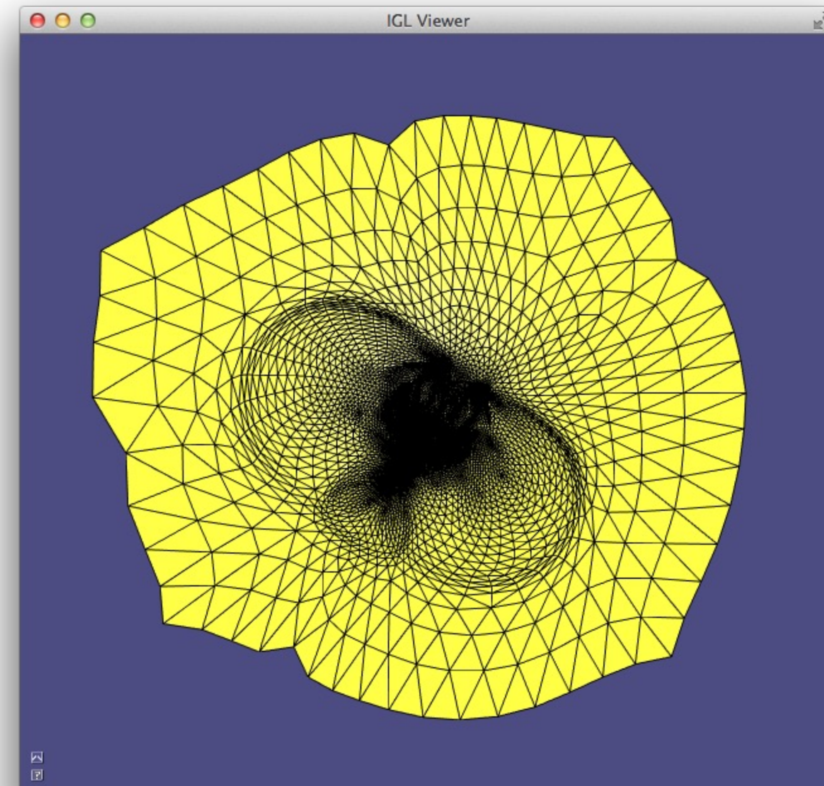
- Doesn't need the entire boundary to be fixed
- Imposing that two vectors on UV maps to 2 orthogonal, same length vectors in 3D.



Least Squares Conformal maps

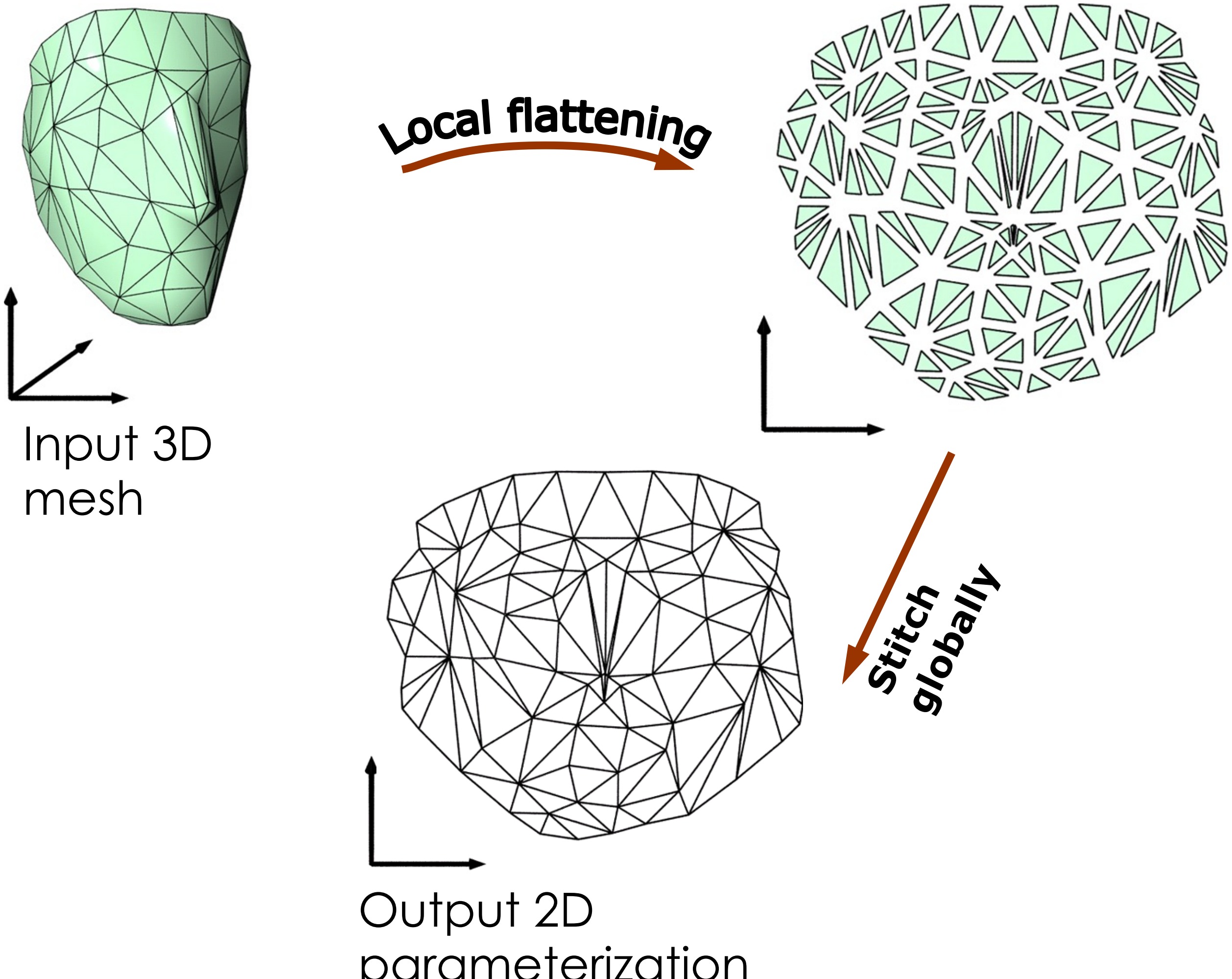
- Need to fix only 2 vertices to disambiguate
- Why?





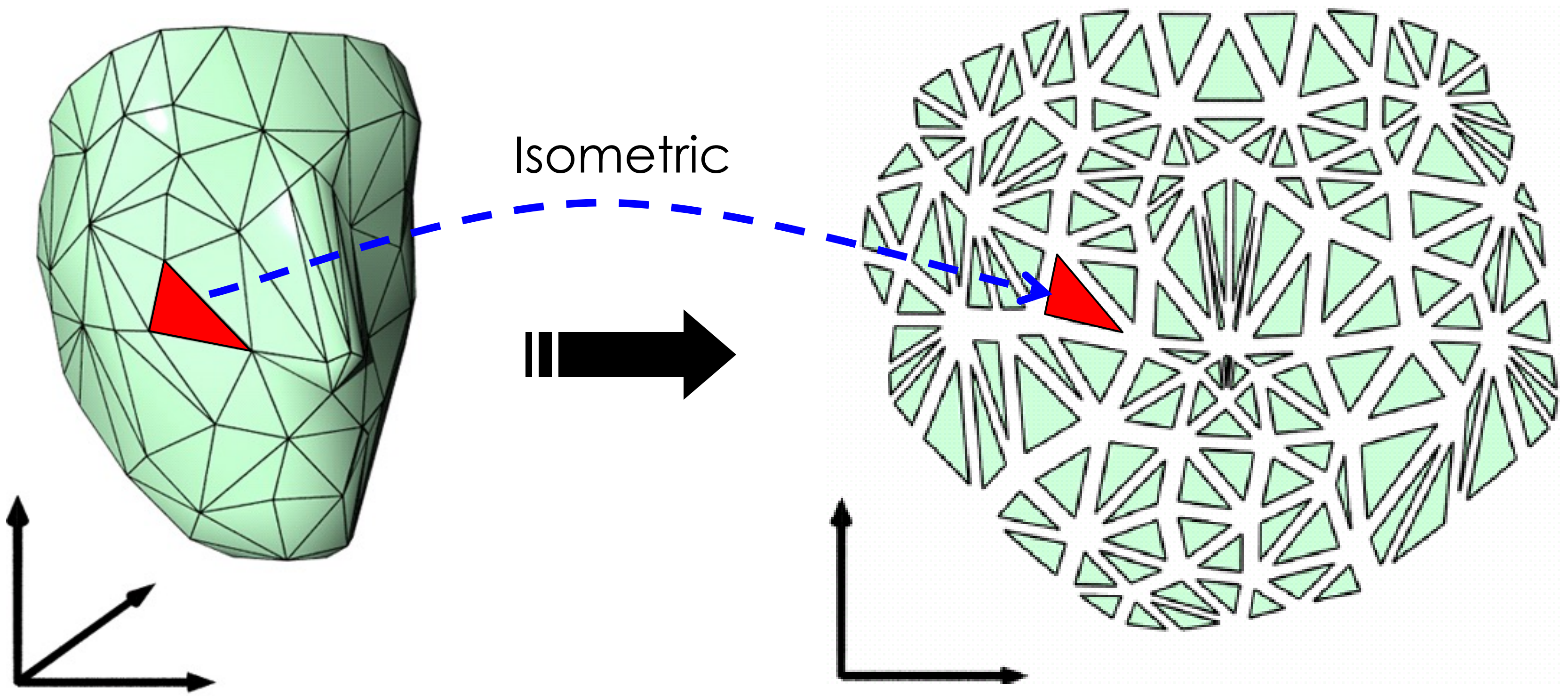
As-rigid-as-possible parametrization (0)

Local-Global Approach



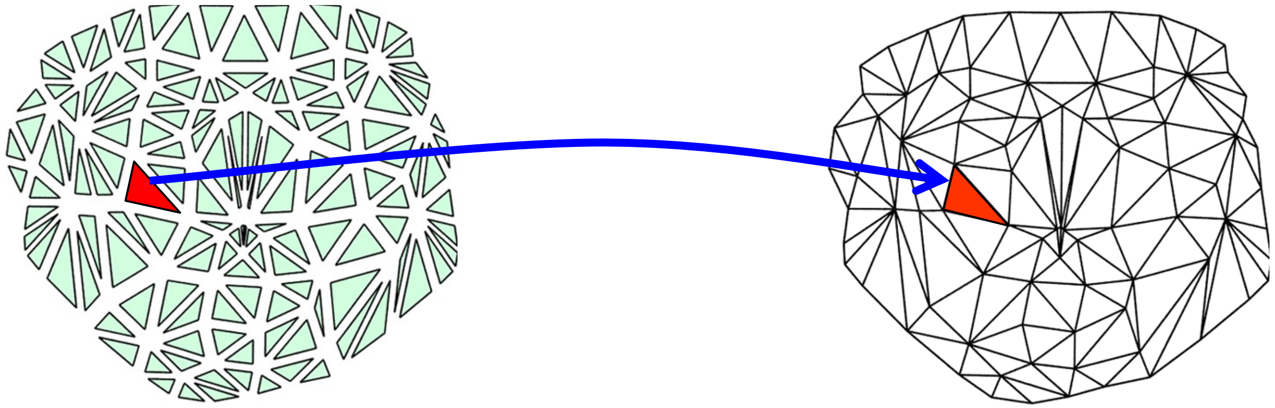
As-rigid-as-possible parametrization (1)

- Each individual triangle is independently flattened into plane without any distortion



As-rigid-as-possible parametrization (1)

- Merge in UV space (averaging or more sophisticated strategied)

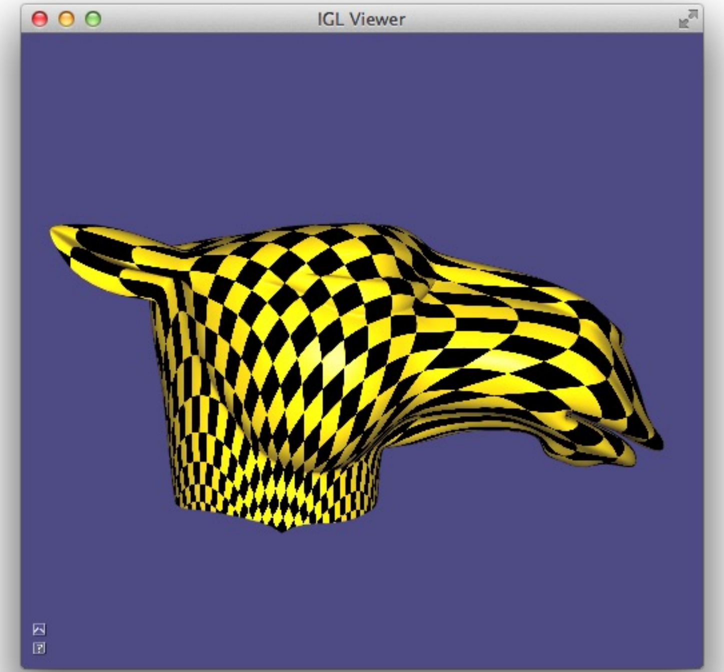


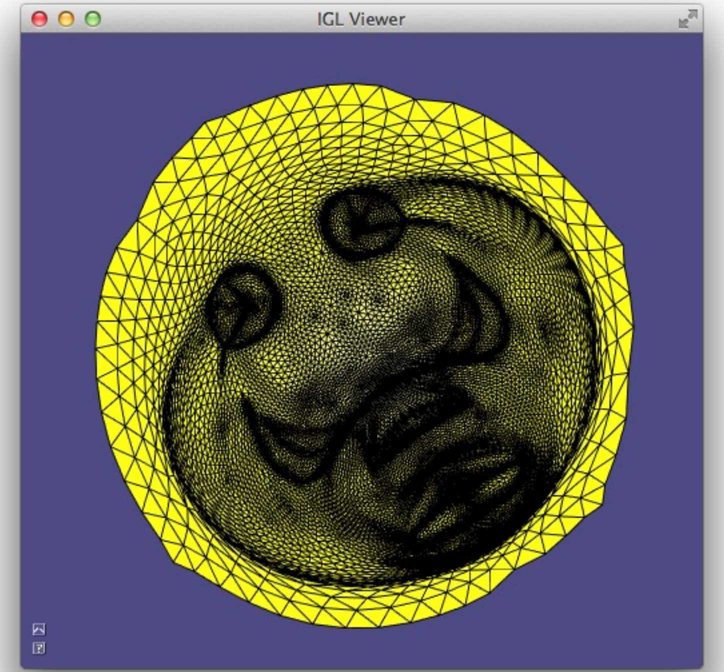
Reference triangles x

Parameterization *u*

As-rigid-as-possible parametrization (1)

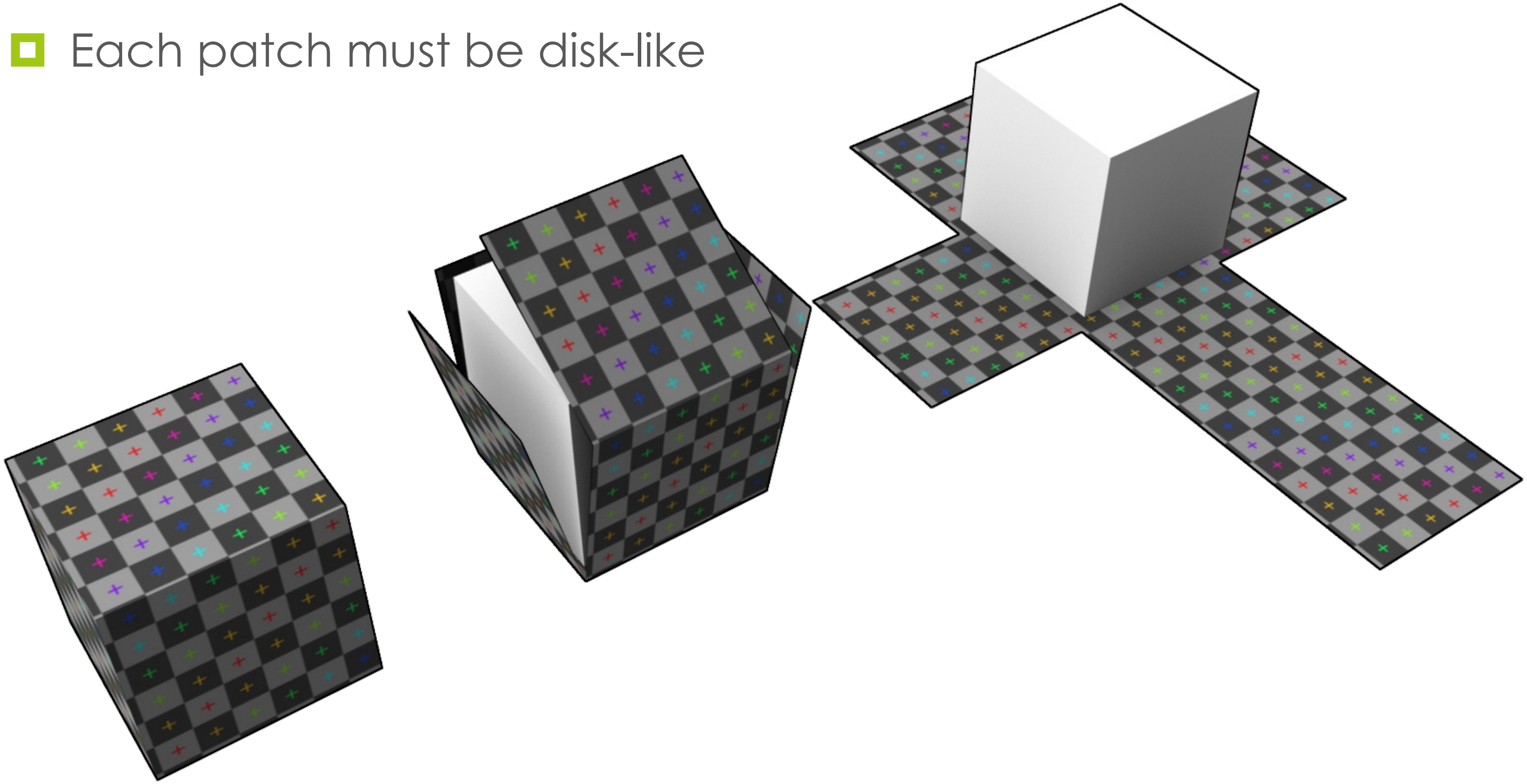
- Warning: it does not guarantee injectivity…





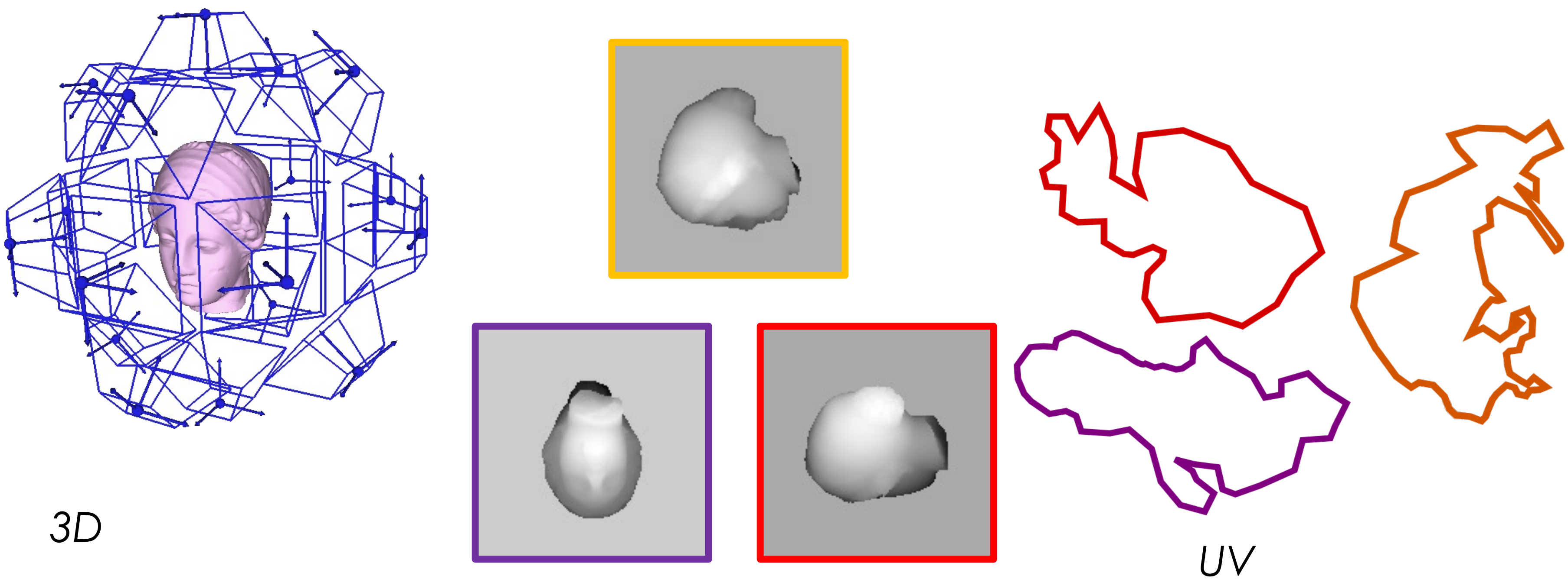
Deriving Cuts

- Splitting the mesh in sub-partitions
- Each patch must be disk-like



Orthoprojection (0)

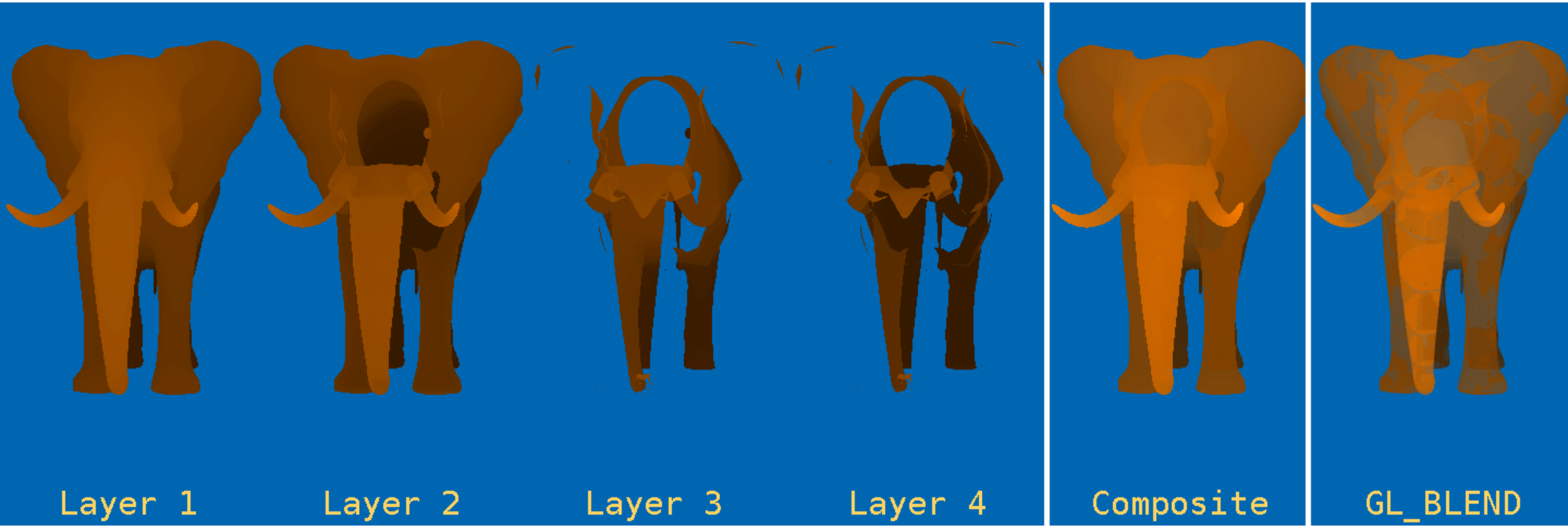
- Use orthographics Projection from multiple directions
- Map each triangle in the “best projection”
- Use depth peeling for handling overlapping parts



Depth peeling

- Depth peeling is **a multipass technique to render translucent polygonal geometry without sorting polygons**.
  (zbuffer and transparency do not work well together)

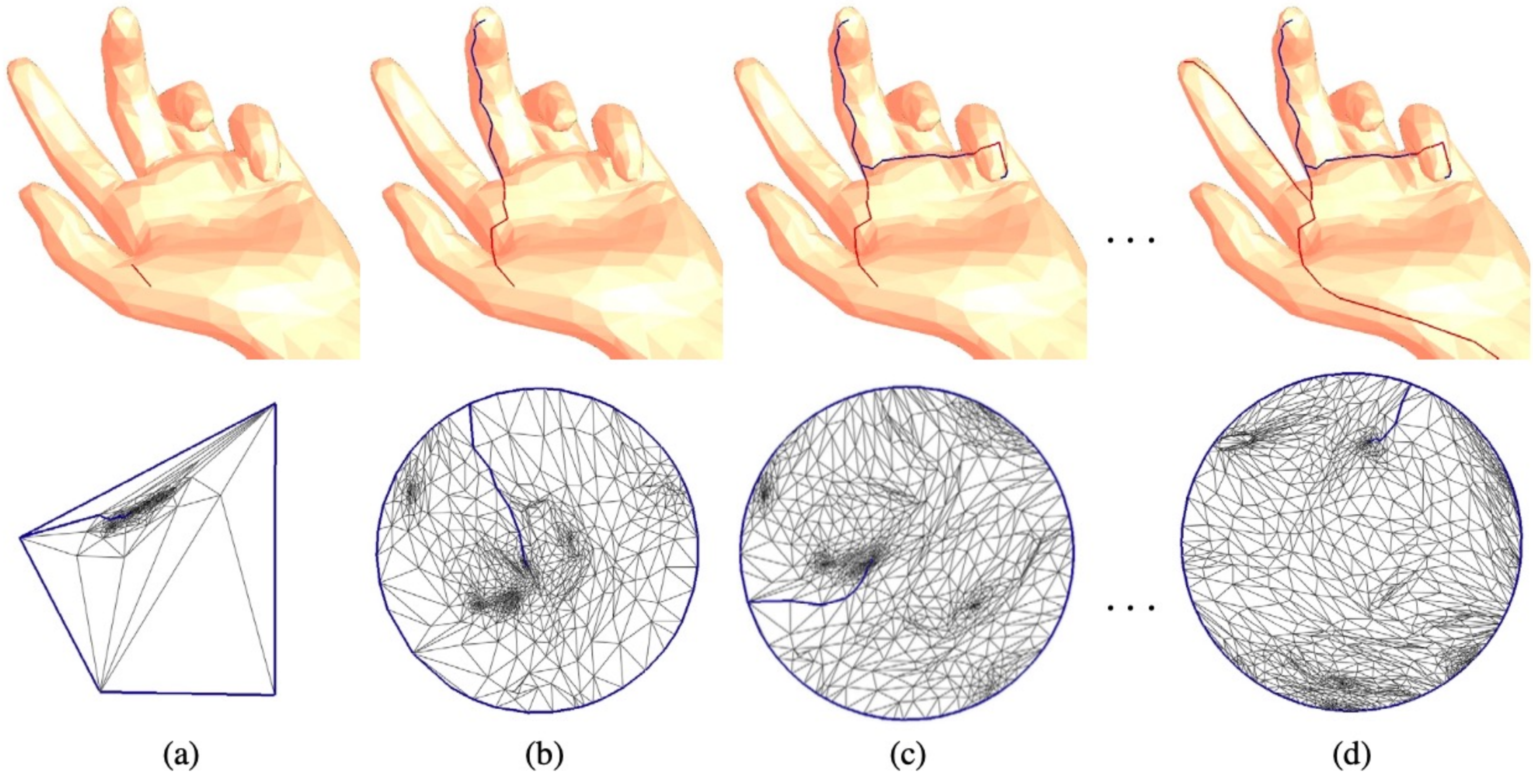
- The idea is to to peel geometry from front to back until there is no more geometry to render.



Orthoprojection (1)

- Small isolated pieces are removed and merged with bigger areas, to avoid fragmentation
- Useful for Color-to-Geometry mapping
- If you have a set of photos aligned over a 3D object they induce a direct parametrization by simply assigning each triangle to the best photo

Growing Cuts



Find the shortest path from the point with the highest distortion to the boundary.
Iterate.

Measuring Parametrization Quality

- Not an easy task to be done in a synthetic way
- Many different measures
  - see *Real-World Textured Things dataset* -> https://texturedmesh.isti.cnr.it/index
- Atlas crumbliness and solidity
  - Crumbliness is the ratio of the total length of the perimeter of the atlas charts, summed over all charts, over to the perimeter of an ideal circle having the same area as the summed area of all charts.