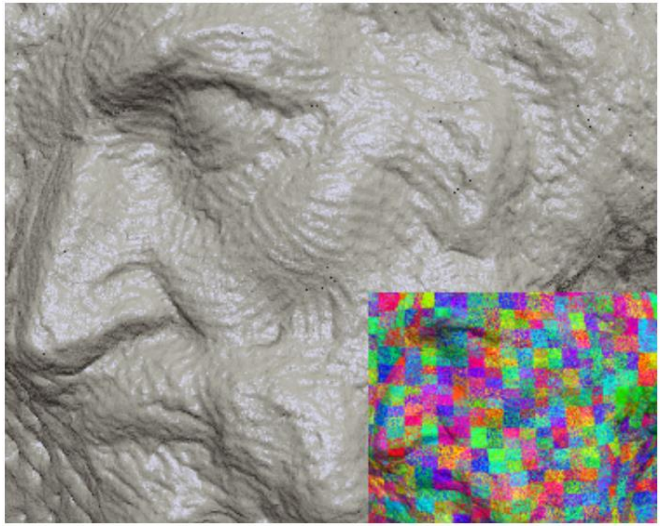
# 3D GEOMETRIC MODELING & PROCESSING
# Sampling

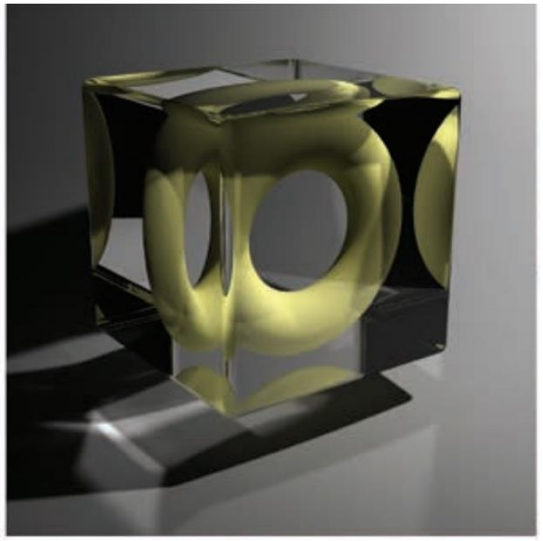
(thanks to Massimiliano Corsini for slides)

# Motivations: Rendering

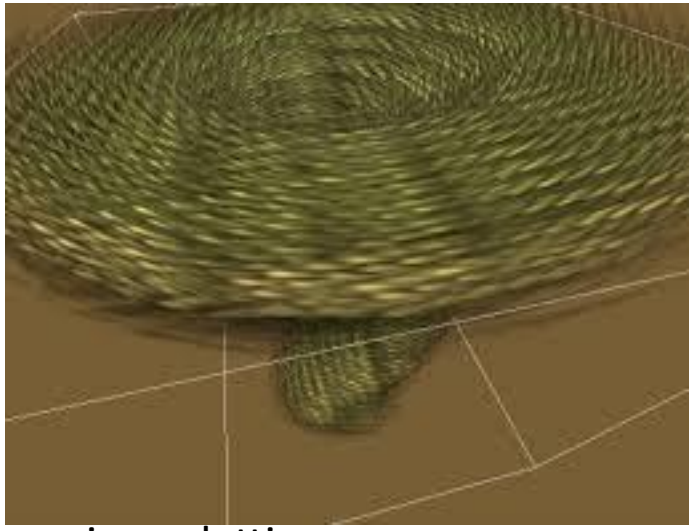
[Layered Point clouds, Gobbetti 2004]



point based rendering

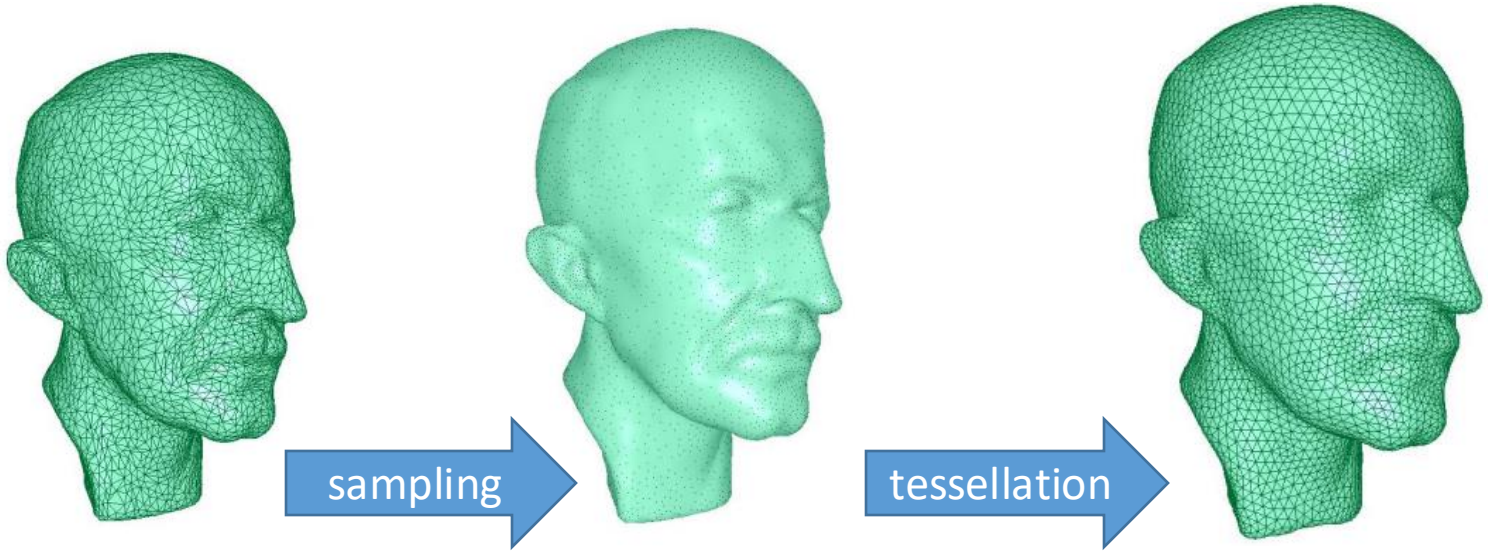


Photorealistic rendering



Gaussian splatting

# Motivations: Remeshing




Input mesh

Point sampling

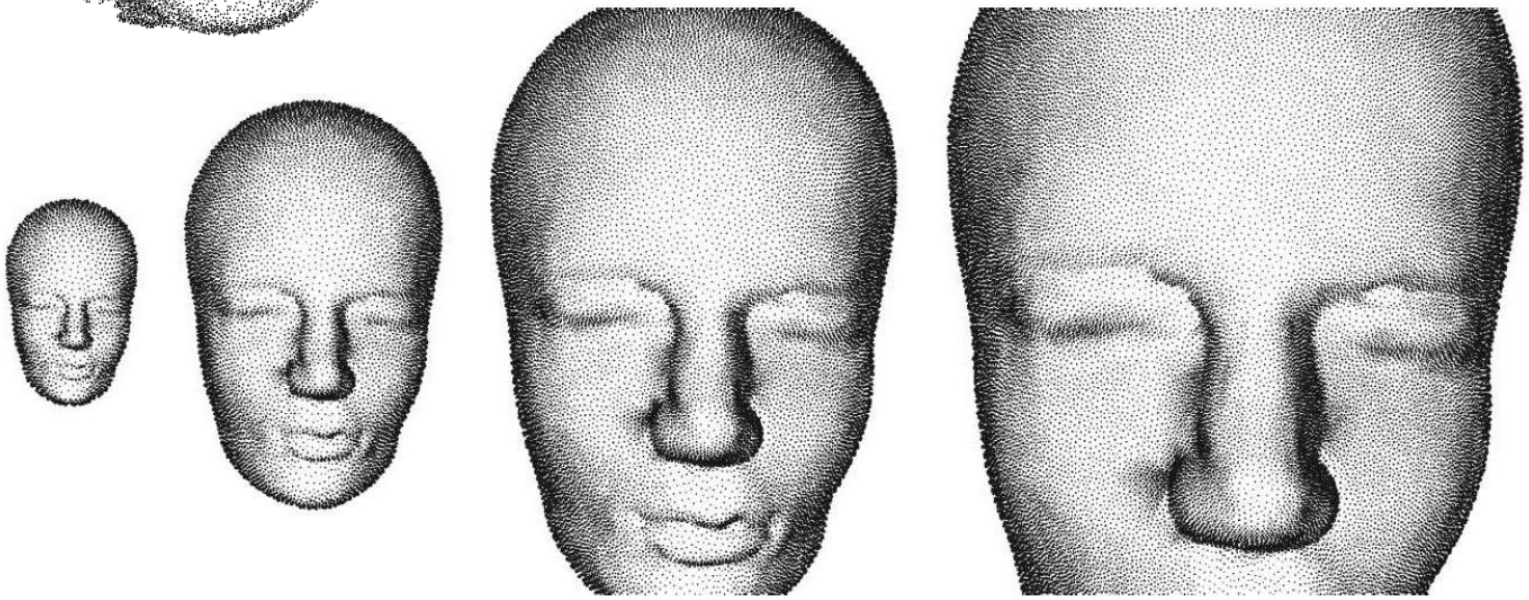
Output mesh

# Motivations: Image/Video stippling

[PixelPie: Maximal Poisson-disk Sampling with Rasterization]

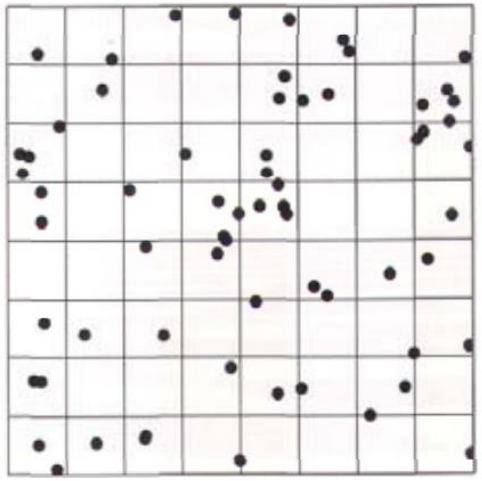
[Bilateral blue noise sampling: Additional algorithms and applications]

[Instant Stippling]



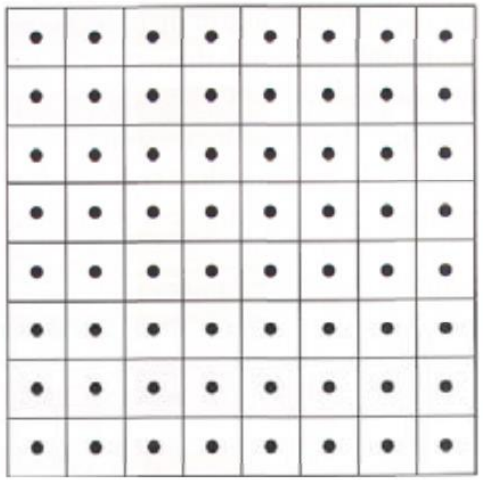
# Jittering

## Random



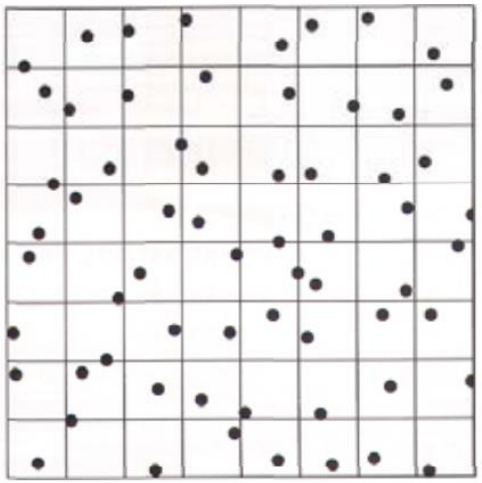
Domain not uniformly sampled

## Uniform



Domain uniformly but not in a random way

## Jittered



Uniform & random. Trading aliasing for noise

```
S JitteredSampling(){
for each Cell in GRID
   S = S + RandomPointInThe Cell()
return S
}
```

# Jittering: from aliasing to noise

*256 samples per pixel as reference*

*1 sample per pixel (jittered)*

*1 sample per pixel (no jitter)*

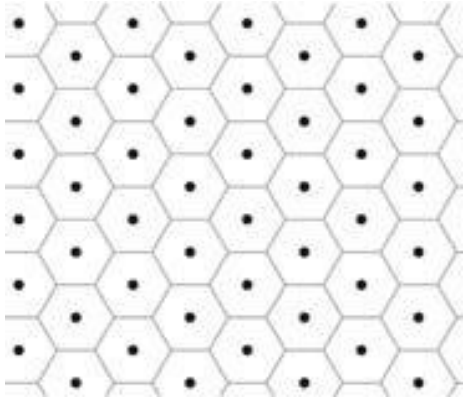
*4 samples per pixel (jittered)*

# How to do sampling

- Characterization
  - Domain: 2D,3D, surfaces..
  - Metric: geodesic, euclidean
  - Shape of the primitive: points, lines, balls
  - Type of algorithms: jittering, dart Throwing, relaxation, Tiling

# Poisson Disk Sampling

- A Poisson Disk Sampling is a sampling $X = \{(x_i, r_i)\ i = 1, \dots, n\}$ such that
  1. Minimal distance: $\forall (x_i, x_j) \in X, ||x_i - x_j|| > \min(r_i, r_j)$
  2. Unbiased sampling: The probability of a region to be covered is proportional to its size
  3. Maximal sampling property: $\Omega \subseteq \cup disk(x_i, r_i)$
- Q: is <u>any</u> PDS a «good» sampling?
- A: No, a maximal PDS sampling (for a given radius) is obtained placing samples at the centers of cells of a hesagonal lattice



# PDS: Dart Throwing [cook86]

```
DartThrowingPDS
n_miss = 0
do{
          x_cand = RandomPoint()
          if  not x_cand ⊆ covered(X) // no disk in X contains x_cand
           X = X + x_cand // hit
          else
           n_miss = n_miss +1 // miss
} while(n_miss / (n_miss +#X) < threshold )
```

- Works on any domain provided a metric
- Very slow convercence rate, $O(n^2)$ asymptotic complexity
- Maximality not guaranteed in given time (likelability of «hit» tends to 0)
- Many approaches devoted to *efficiency* and *maximality*

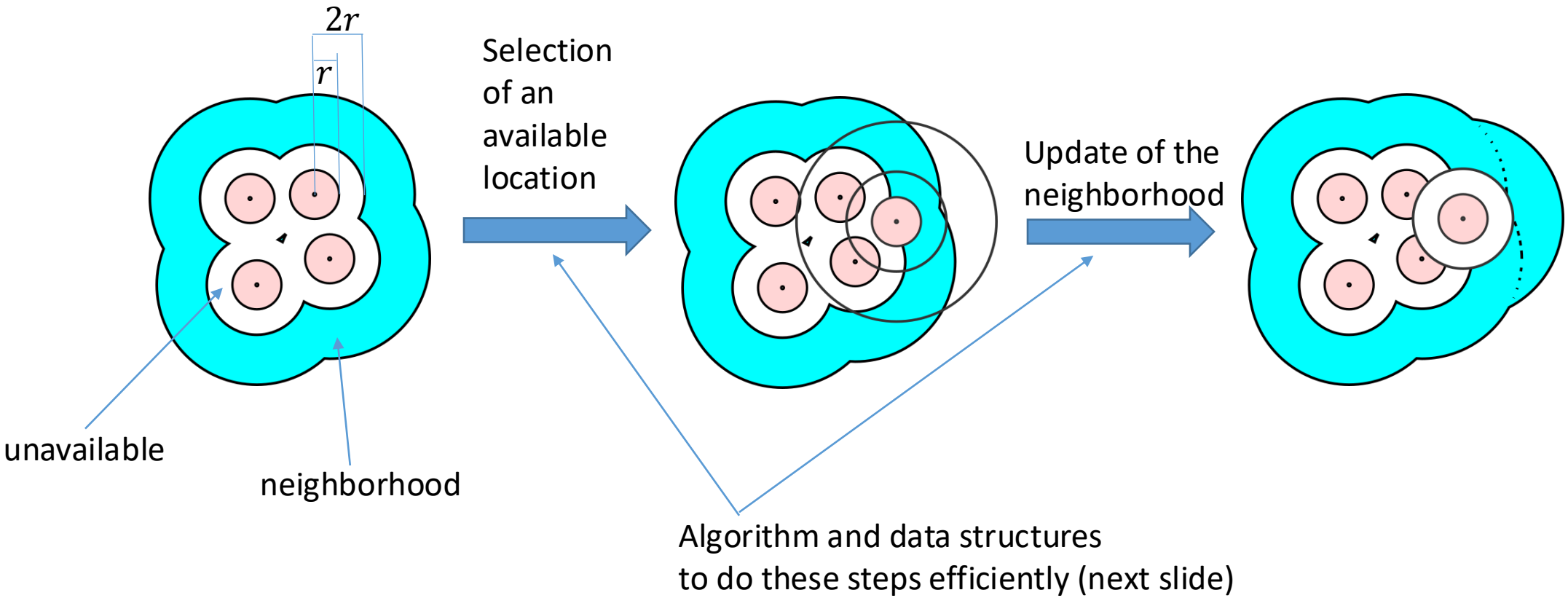
# Efficiency in PDS algorithms

- Two basic operations that determine convercence speed of a PDS algorithms

1. choosing a sample location with unbiased probability
2. Testing if a new location is not already covered

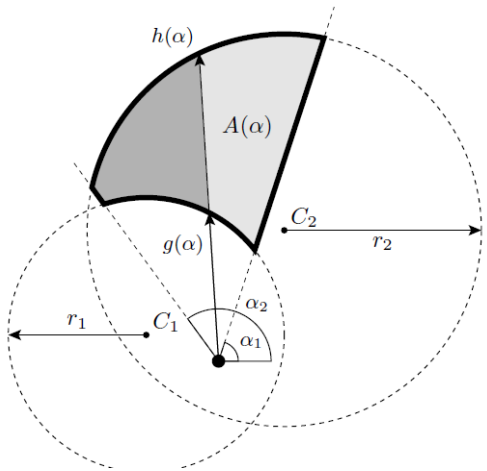
# Scalloping [Dunbar06](1/4): Dart Throwing in $O(n \log n)$

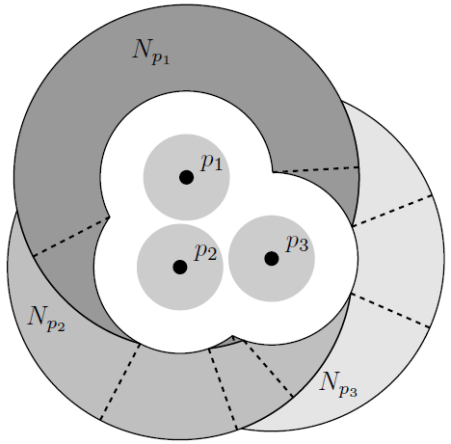
- Core idea: if a sampling is not maximal, there must be an *available* location in the neighborhood of the *unavailable* region



# Scalloping (2/4): Dart Throwing in $O(n \log n)$

- *Scalloped regions:*
- *Def. Scalloped sector:* a region of the domain bounded by two circular arcs
- Def: *Scalloped region*: a union of scalloped sectors



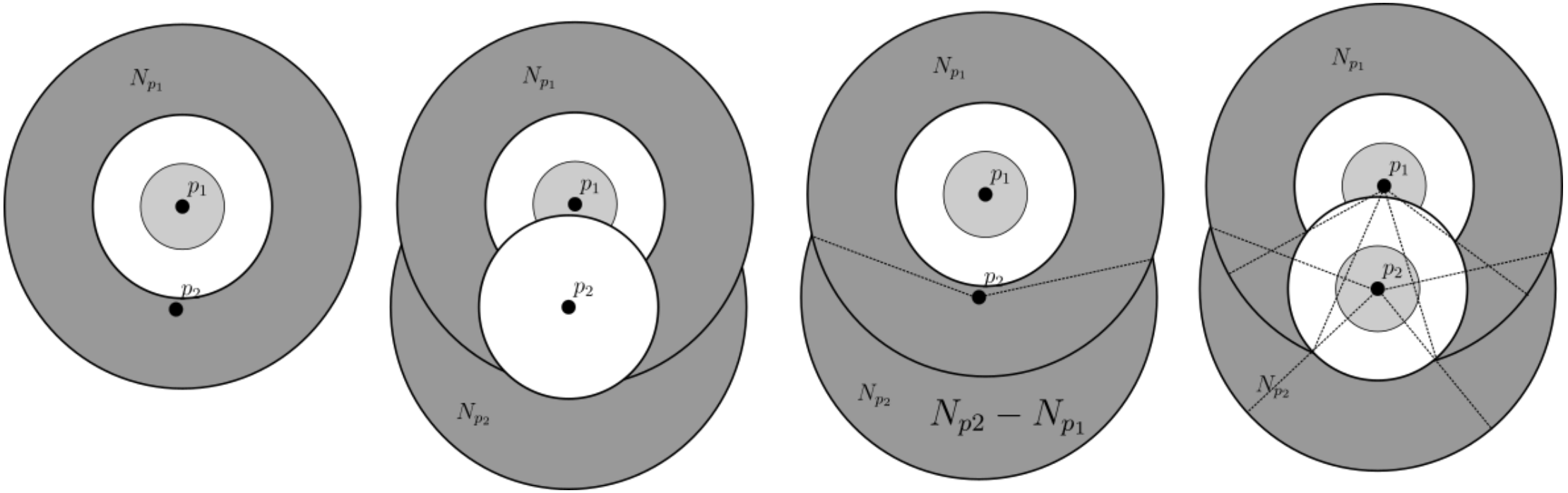
# Scalloping (3/4): adding a new disk

Order of insertion

$$N_p = D(p, 4r) - \bigcup_{p' \in P} \begin{cases} D(p', 4r), & p' < p \\ D(p', 2r), & p' \geq p \end{cases}$$

Available neighborhood

disk

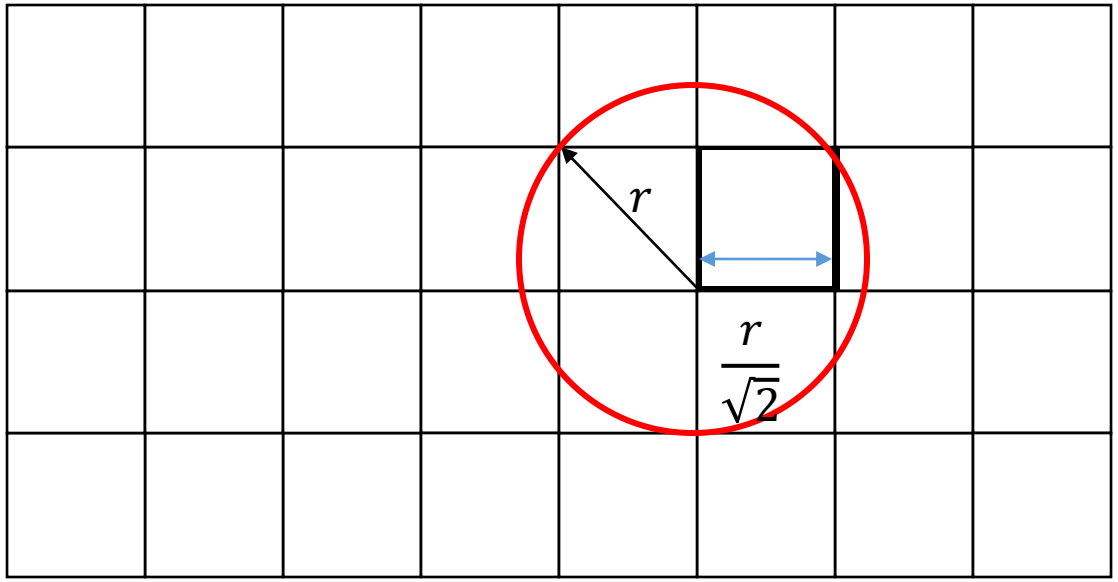


# Scalloping (4/4): Speed

- The maximum number of scalloped sectors of r neighboorhod is bounded by a constant
- The update of neighboorhood is limited to the 4r radius from the sample and can be done in O(1)
- Unbiased sampling. All available neighborhoods are stored in a balanced tree O(log N)

# Hierarchical Dart Throwing [White07] (1/3)

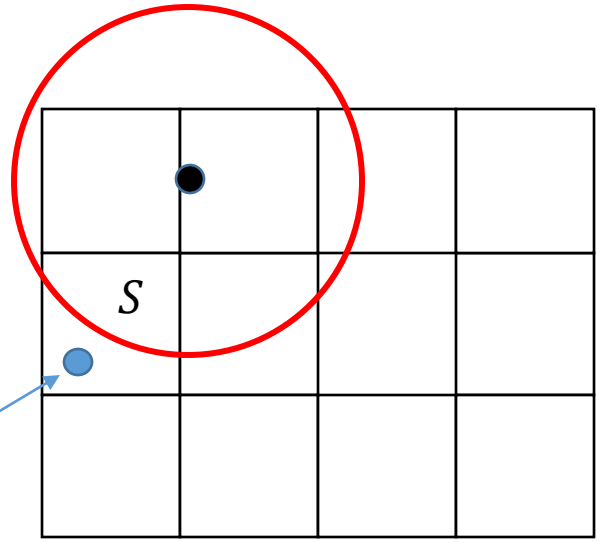
- Regular grid where each cell is the root of a quadtree
- Size of each cell so that it is completely covered by a disk which center is inside the cell
- **def**: *Active cell*
  Cell not yet entirely covered by a disk
- **def:** *Active list with index **i***
  *List of active cells at level i*
- *Initial condition: Active list L0 contains all the cells of the grid*

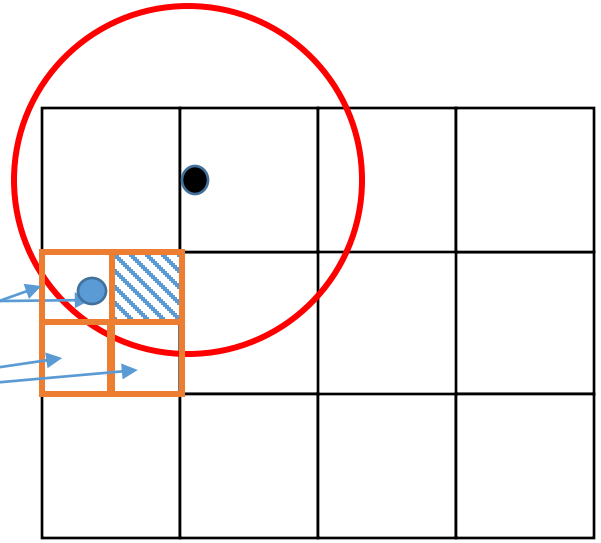


# Hierarchical Dart Throwing (2/3)

**Hierarchical Dart Throwing**

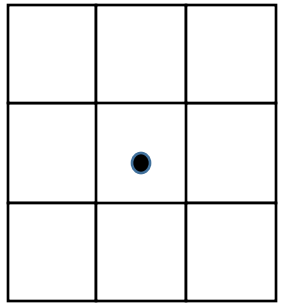
Put base level squares on active list 0 (the base level).
Initialize the point set to be empty.
While there are active squares
  Choose an active square, $S$, with prob. proportional to area.
  Let $i$ be the index of the active list containing $S$.
  Remove $S$ from the active lists.
  If $S$ is not covered by a point currently in the point set
    Choose a random point, $P$, inside square $S$.
    If $P$ satisfies the minimum distance requirement
      Add $P$ to the point set.
    Else
      Split $S$ into its four child squares.
      Check each child square to see if it is covered.
      Put each non-covered child of $S$ on active list $i+1$.



# HDT (3/3 ): Cost

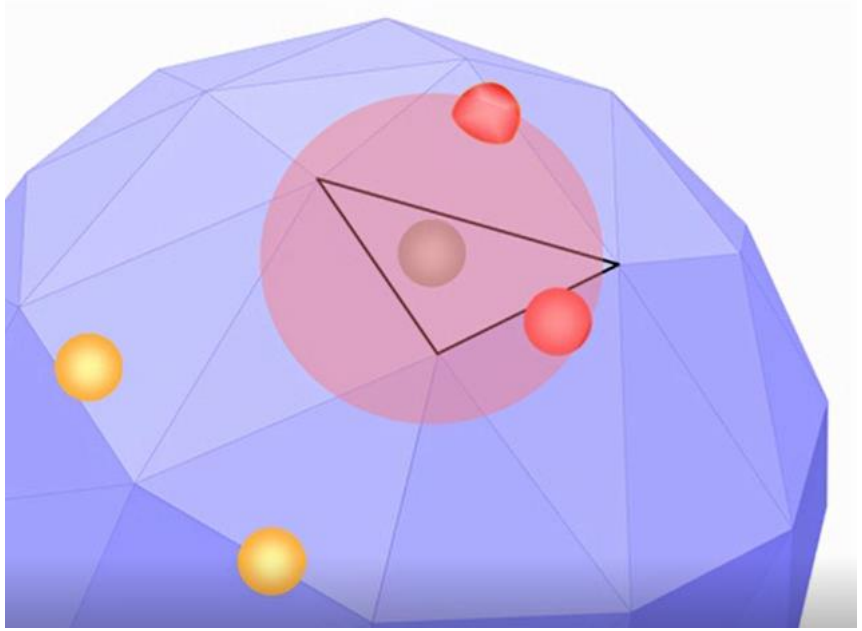
- Coverage test: a secondary uniform grid of cells with size $r$ stores a copy of the point set. By construction, no cell can contain more than 4 points. Therefore, the cost for coverage is constant
- Choice of sample:
  1. keep the sum of areas for each list $a_0, \dots, a_k$
  2. Generate a random number in [0,1]
  3. Find m s.t. $\sum_{i=0}^{m-1} a_i \leq a_{tot} x < \sum_{i=0}^{m} a_i$
  4. Pick a random square in the list



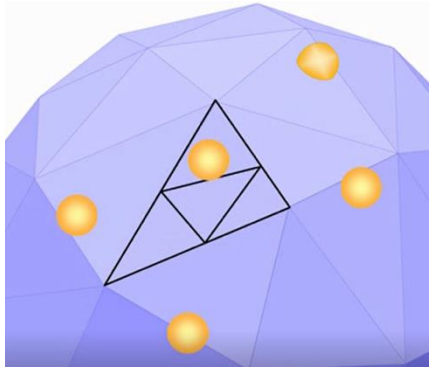
1. Asyntoptic cost: how many cells are created? Authors claim $O(N)$: «*While we cannot provide a rigorous proof, we are convinced that hierarchical dart throwing is O(N) in both space and time on average*»

# PDS on surfaces [Cline09] (1/6)

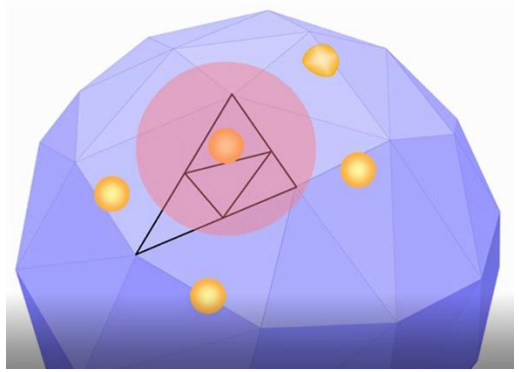
- «Essentially» the same as HDT
  - Replace «uniform grid» with «triangulation»
  - Replace «quadtree» with 1-4 triangle subdivision



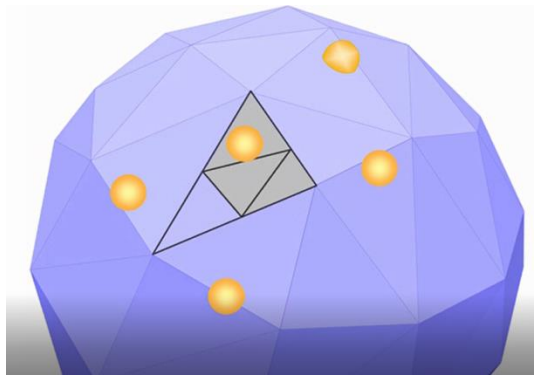
# PDS on surfaces [Cline09] (2/6)



Pick a point randomly, check that is not closer than r to any other point



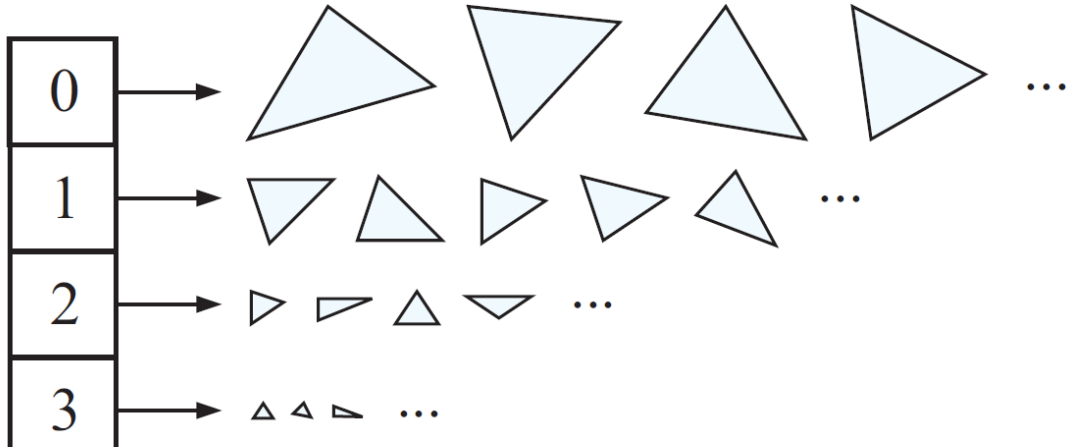
until the triangle containing the point is not entirely within distance r, iterate 1-4 splitting



Remove entirely covered triangles from the activel list.
End when the list is empty.

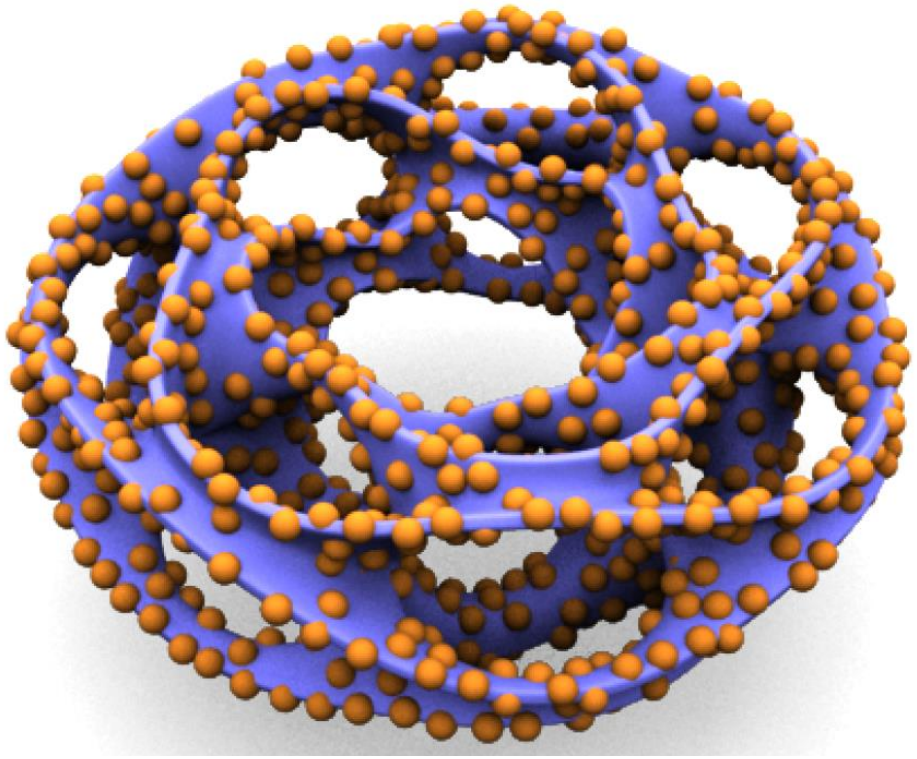
# PDS on surfaces [Cline09] (3/6): Cost

- Coverage test. Constant like in [White07]
- Unbiased sampling: Logarithmic binning
  - each bin store a list of triangles within a range area
  1. a = random value in [0,total_area]
  2. Linear search on the bins until sum_b>a
  3. Pick a triangle and accept with prob triangle_area/bin_area. Repeat until accept
  4. Pick a random point in the triangle



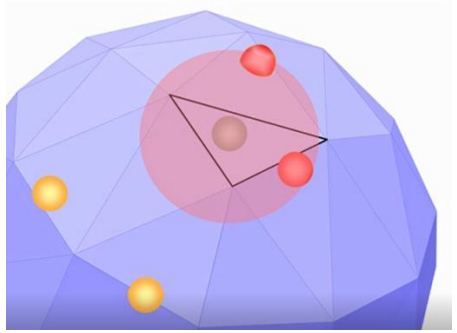
# PDS on surfaces [Cline09] (4/6)

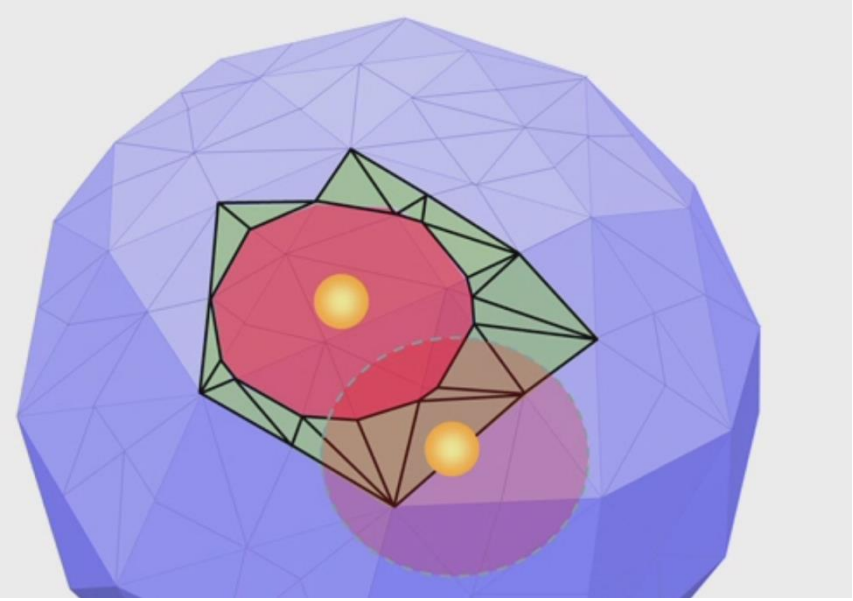
- Does it work well?



# PDS on surfaces [Cline09] (5/6)

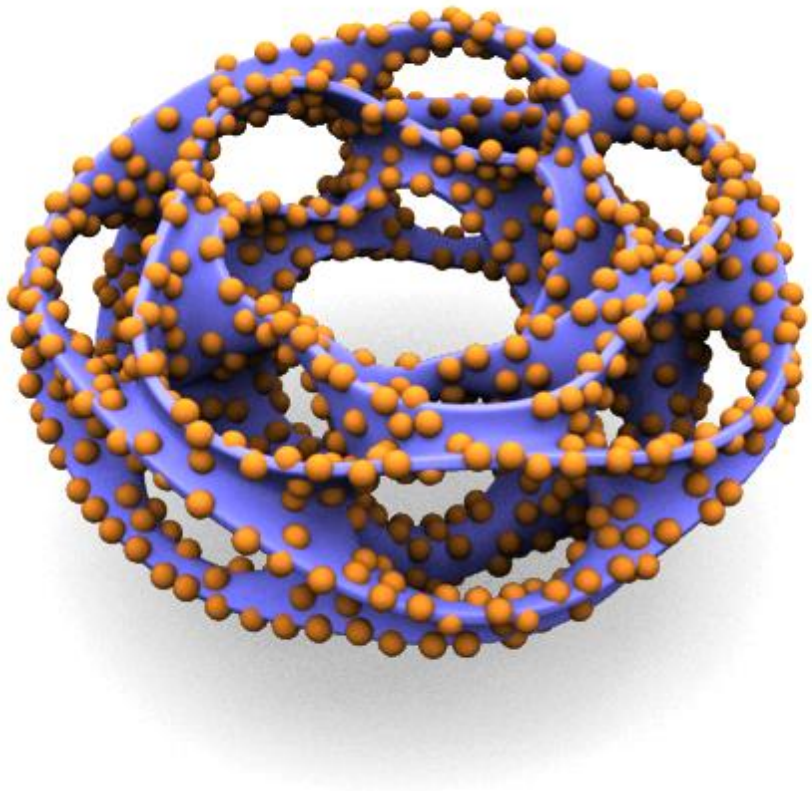
- «Essentially» the same as HDT
  - Replace «uniform grid» with «triangolation»
  - Replace «quadtree» with 1-4 triangle subdivision
  - Replace «Euclidean distance» with «geodesic distance»



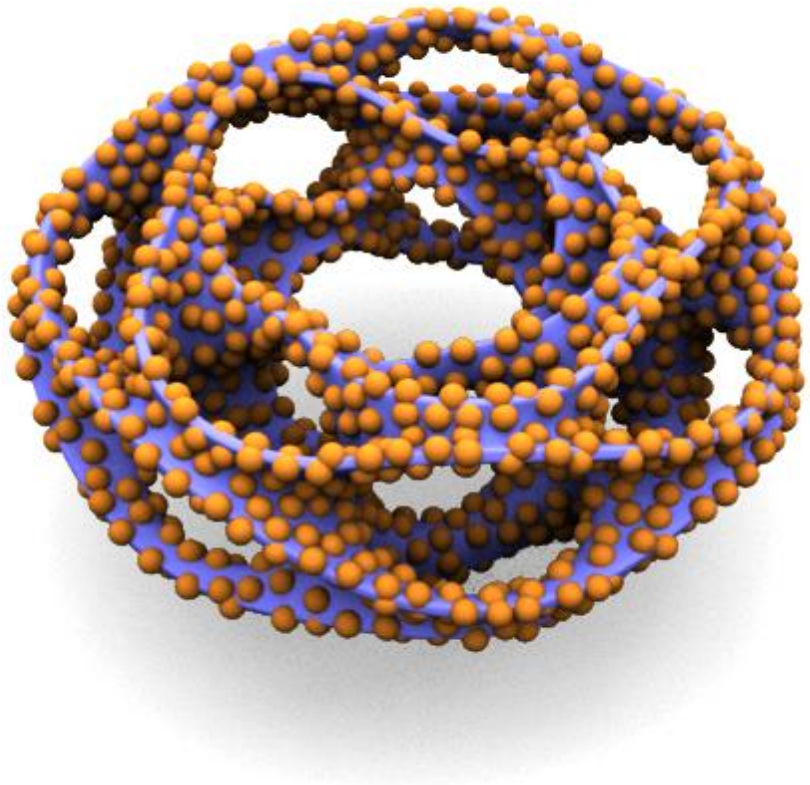


# PDS on surfaces (6/6)

Euclidean distance

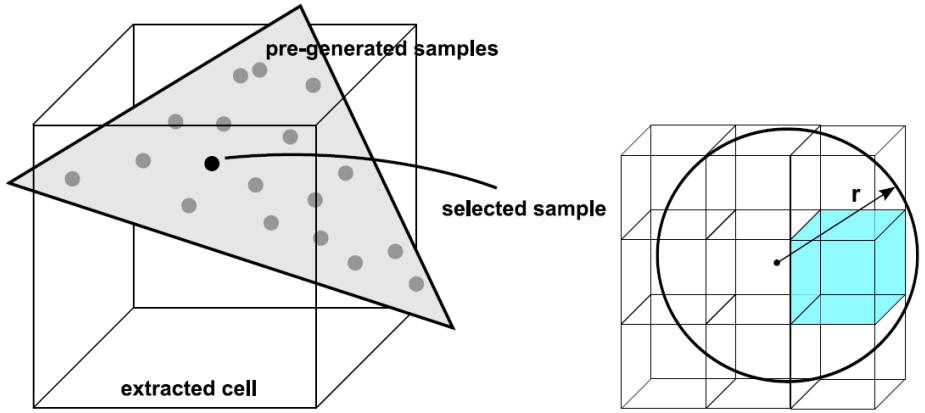


geodesic distance



# HDT in 3D space [Corsini12]

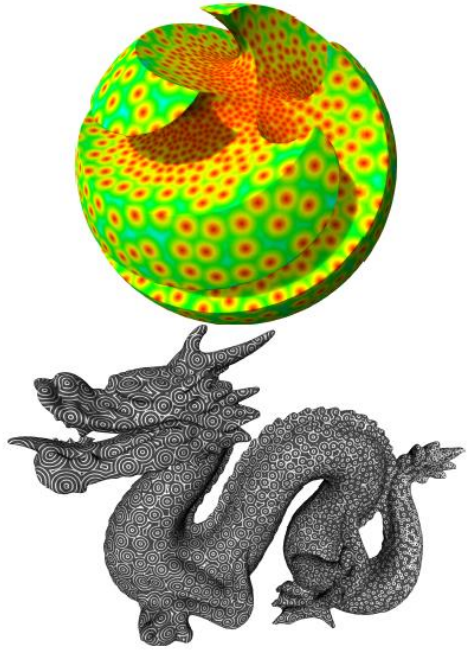
- Surface immersed in a 3D uniform grid where each cell is the root of a oct-tree
- The data structure is a direct extension of HDT except that the cell **is not the** domain but it **contains** the domain (the triangles)
- Basic operation: once an active cell is chosen, pick a point on the contained surface
- Use of *pregenerated samples*



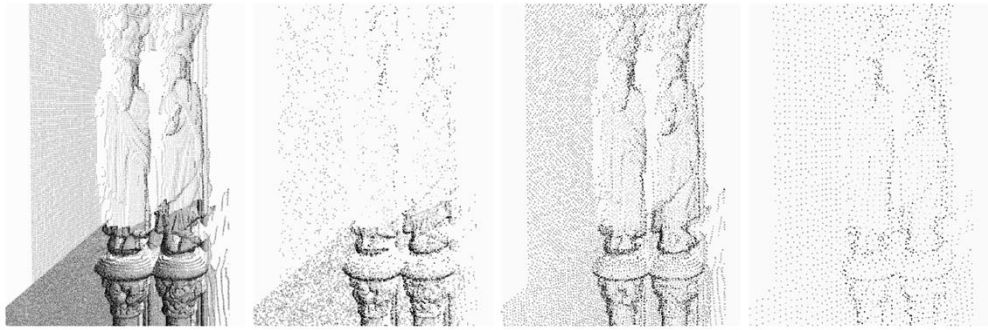
# Samples generation

- Variant: just shuffle the initial overasampling and iterate point removal. No hierarchy needed anymore!
  1. Generate the pool: a dense sampling of the surface
  2. Until there are available samples
     1. Pick a sample randomly
     2. Remove the sample closer than he disk radius from the pool
- Q: Both versions work well: where is the trick?
- A: the creation of the pool essentially converts the initial «continuous» domain (faces) in a point sampled one. The algorithm cannot guarantee maximality of the sampling up to the intersample distance of the initial pool

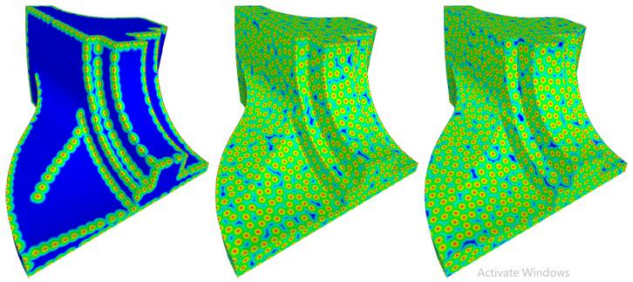
# Example uses



Importance sampling / variable radius



Subsampling Point clouds



Edge preserving sampling

# Voronoi Decomposition

# A recurring pattern in Nature

# A recurring pattern in Nature

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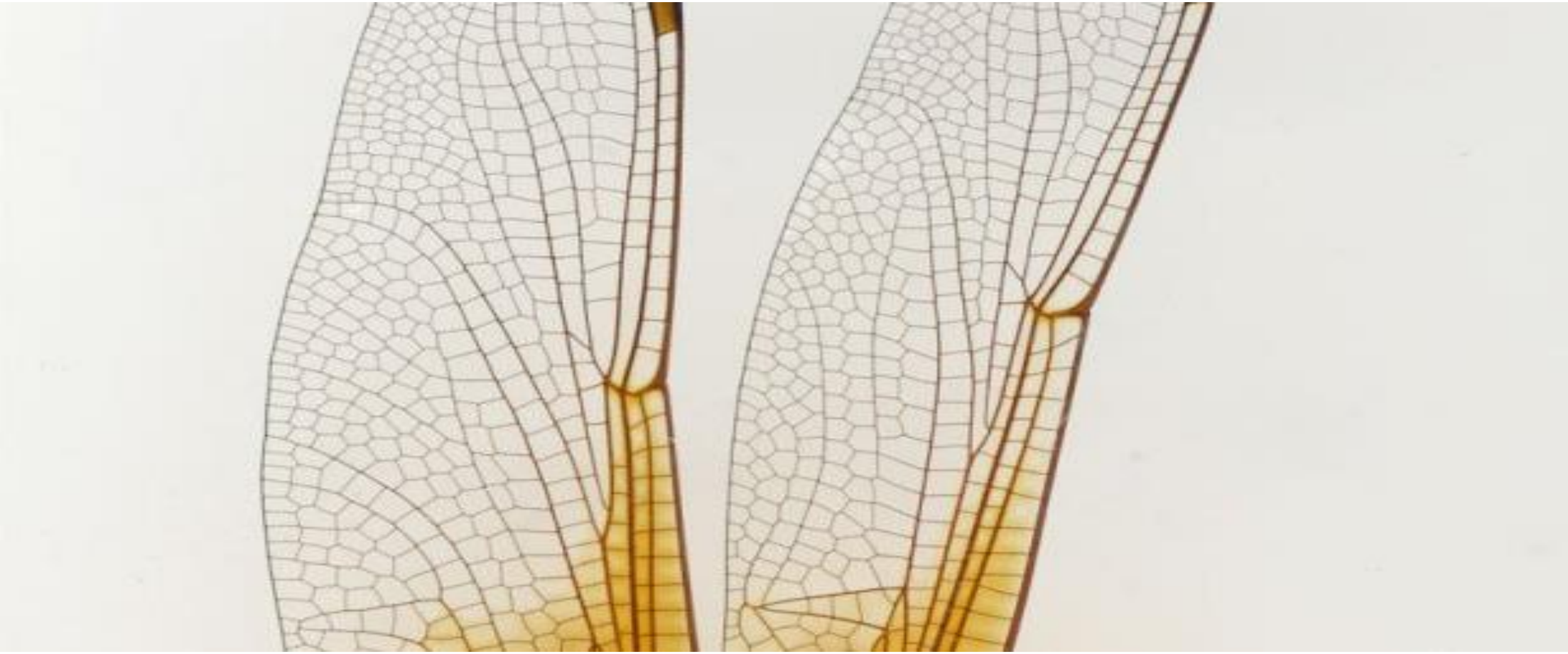
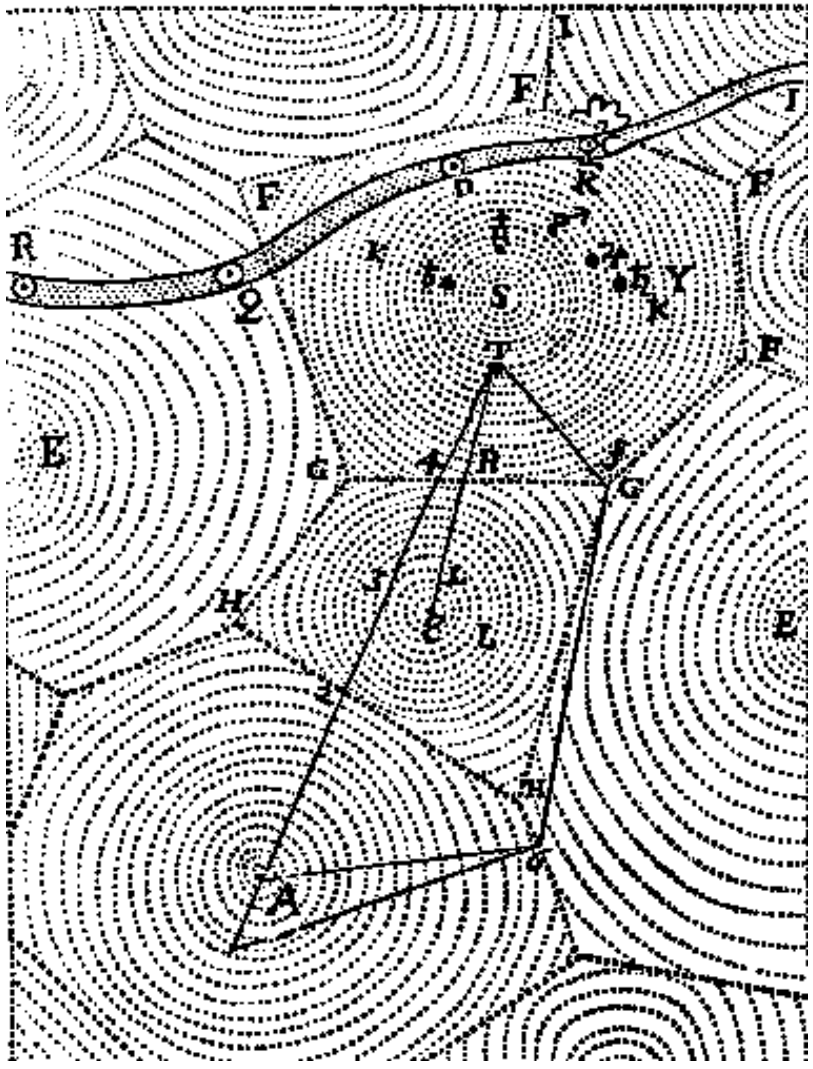
# A recurring pattern in nature



# A recurring pattern in Nature

# Voronoi Diagrams

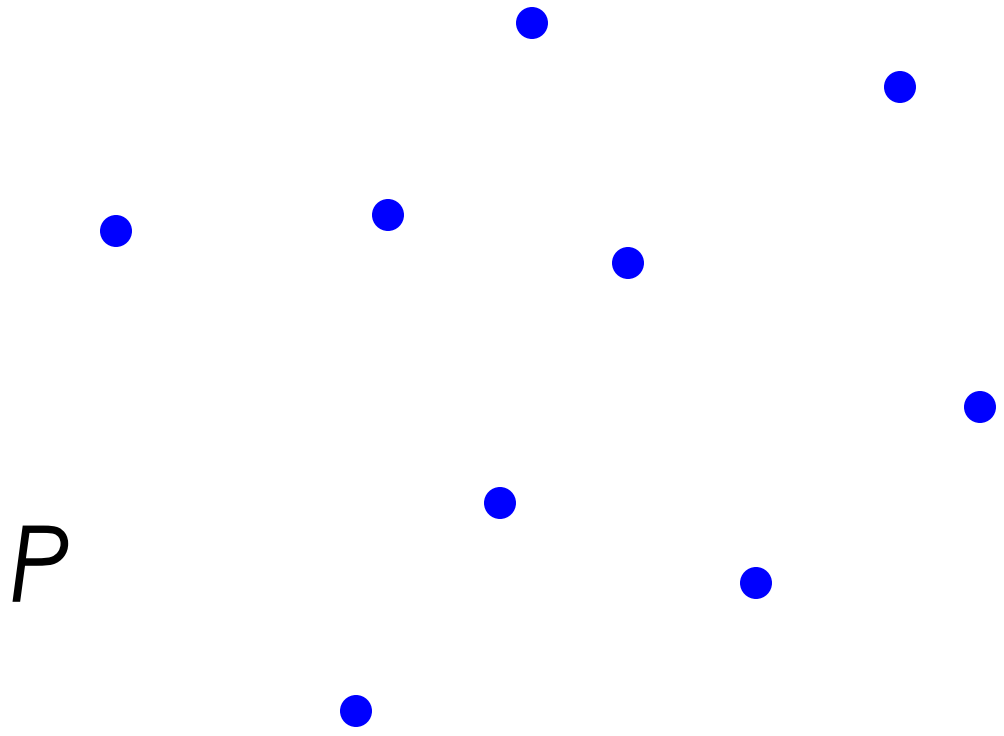
- Main idea: a discrete set of entities competing for resources
- Post office problem
- Discussed since:
  - René Descartes, Le Monde (1644)



# Voronoi: a bit of theory

Consider a set $P$ of points on the plane

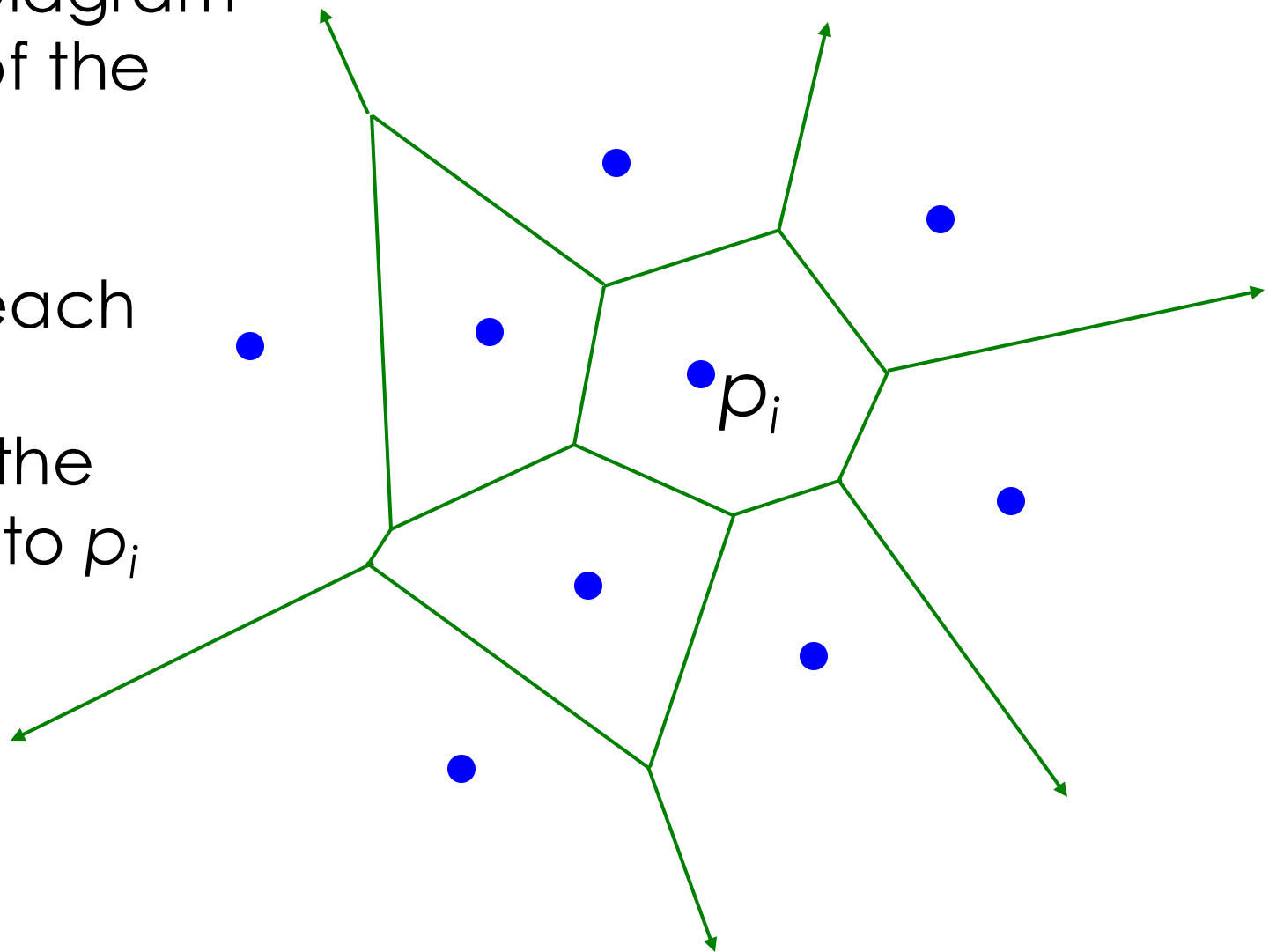
$P = \{p_0, \ldots, p_i, \ldots, p_n\}$



# Voronoi: a bit of theory

The Voronoi Diagram is a partition of the plane

Consider for each point $p_i$
the region of the plane closest to $p_i$

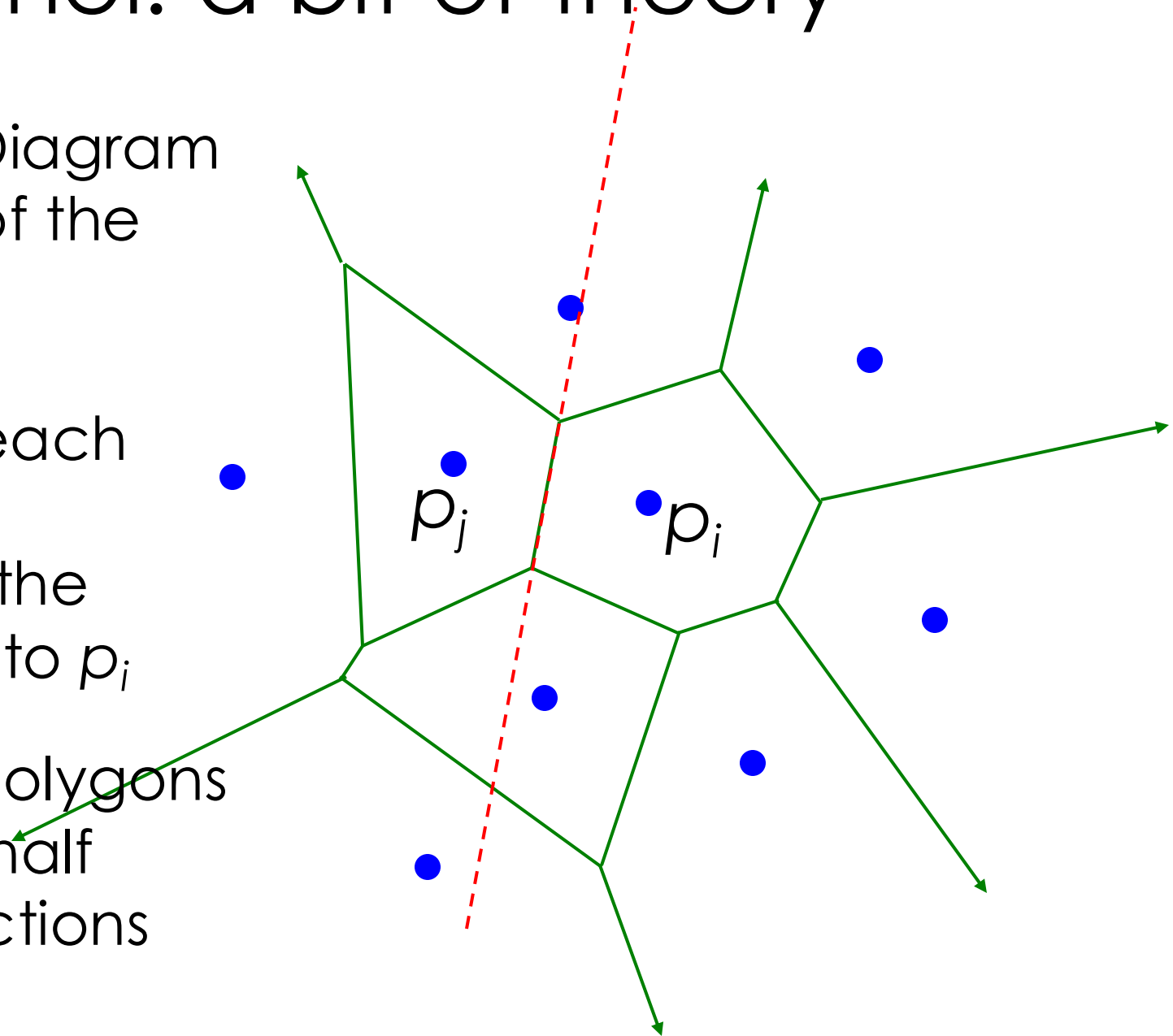


# Voronoi: a bit of theory

The Voronoi Diagram is a partition of the plane

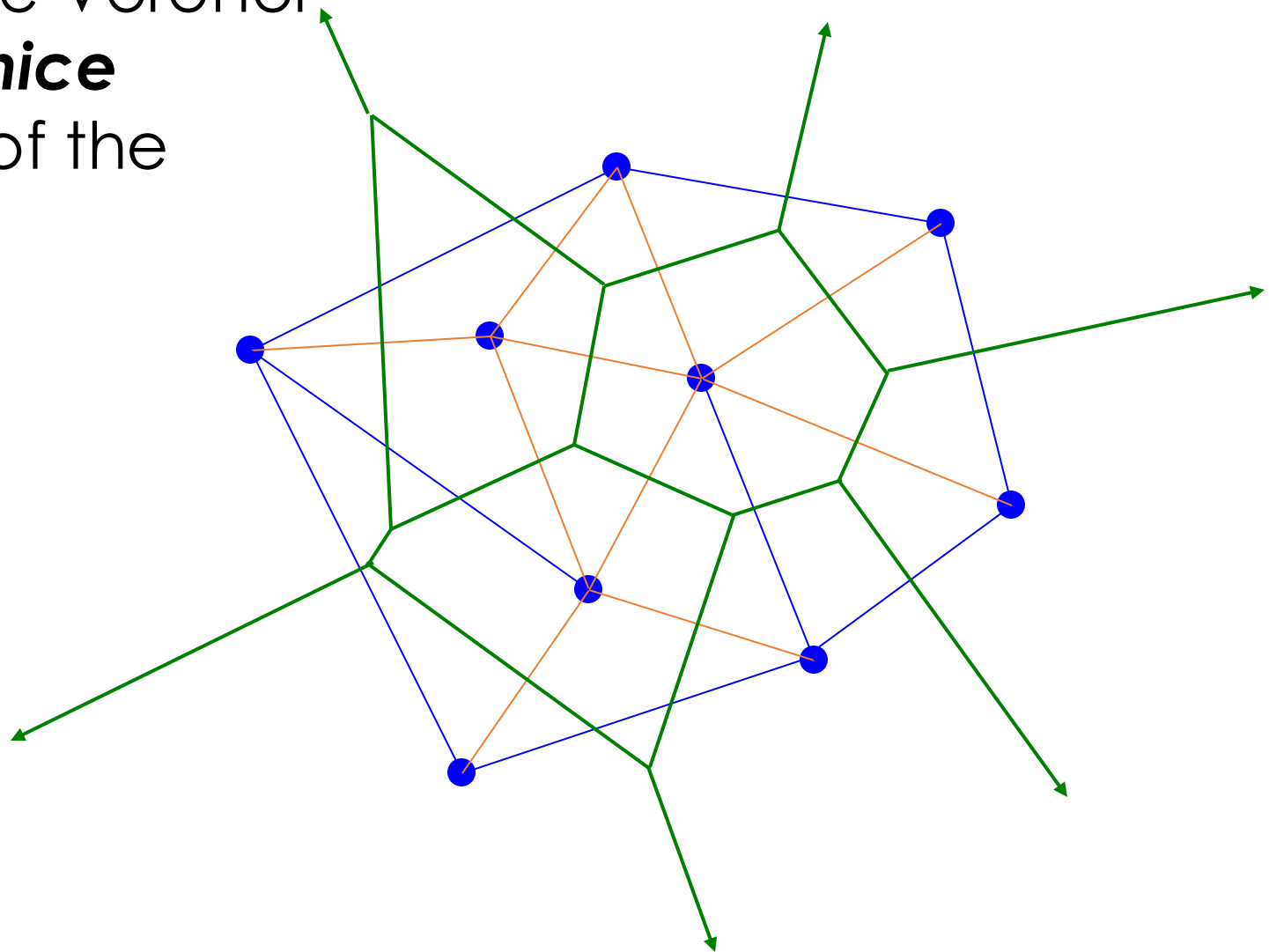
Consider for each point $p_i$
the region of the plane closest to $p_i$

Regions are polygons bounded by half plane intersections



# Voronoi: a bit of theory

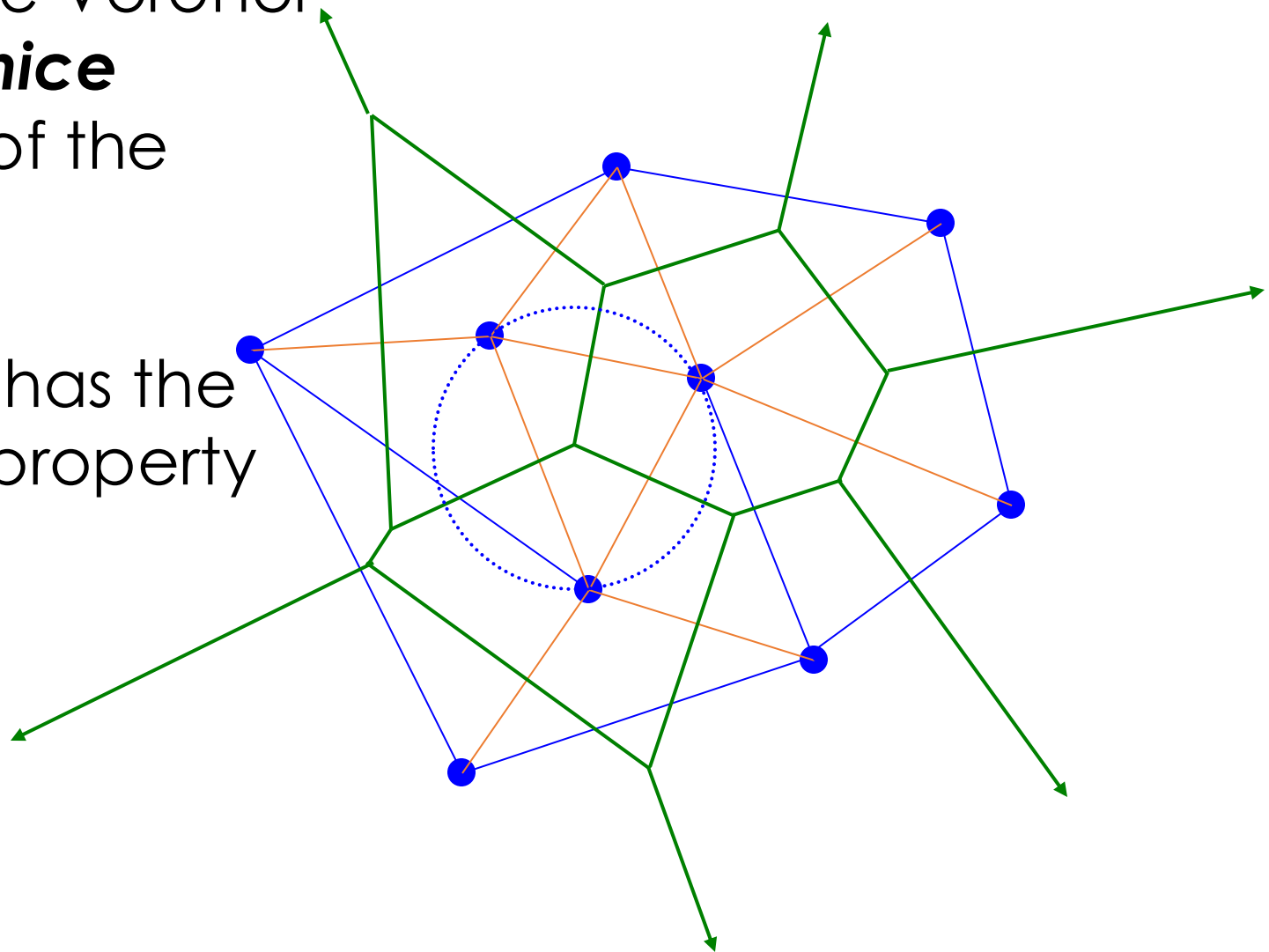
The dual of the Voronoi Diagram is a ***nice*** triangulation of the point set



# Voronoi: a bit of theory

The dual of the Voronoi Diagram is a ***nice*** triangulation of the point set

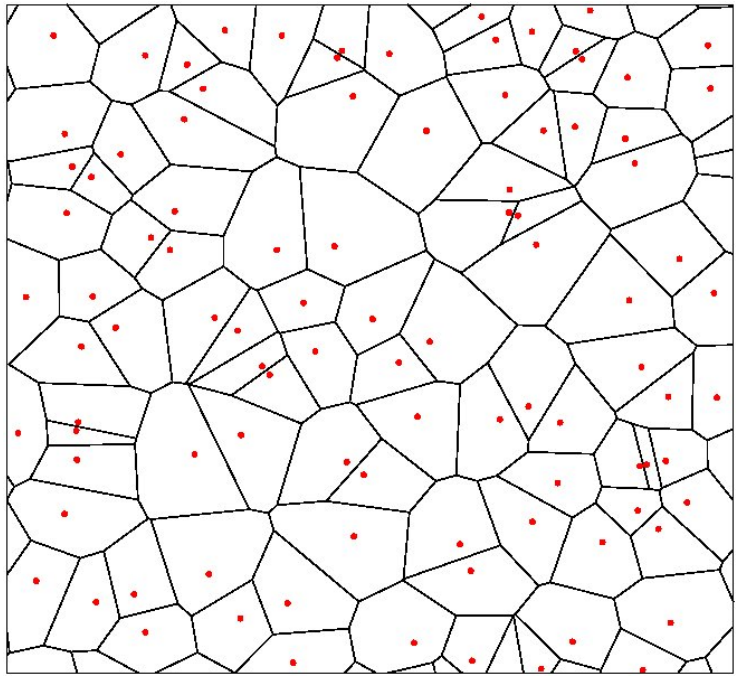
Each triangle has the *empty-circle* property

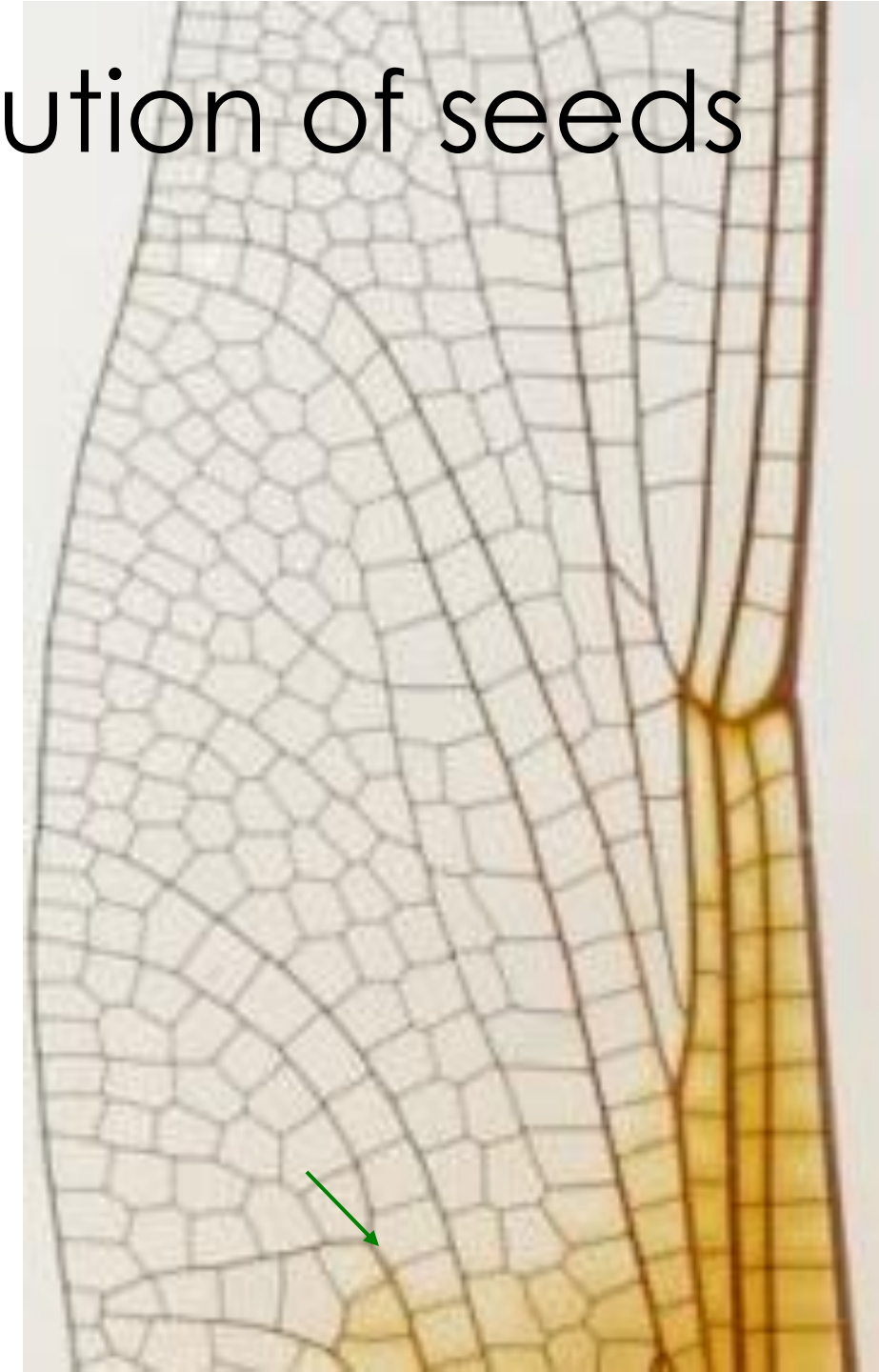


# Voronoi: Distribution of seeds

The Voronoi Diagram is a partition of the plane

**Nice** Voronoi Diagrams came from **nice** point distributions
How to get nice point sets?



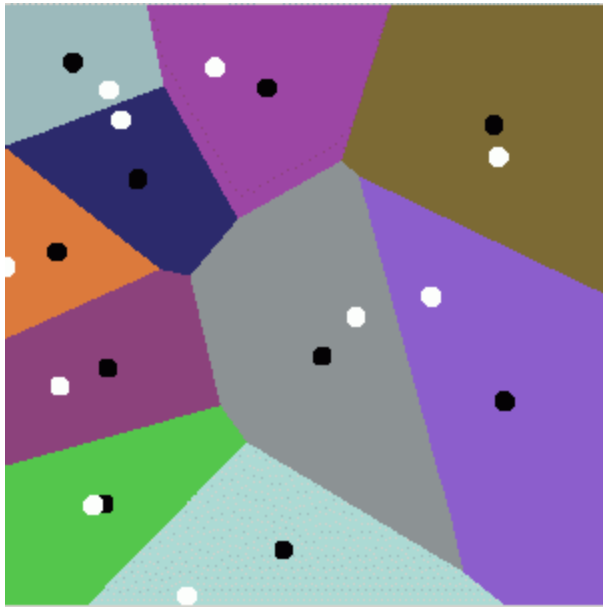


# Centroidal Voronoi Diagrams

- VD generated from random seeds are not well placed around the seeds.
- Region are not “centered” around the seeds

# Centroidal Voronoi Diagrams

- VD generated from random seeds are not well placed around the seeds.
- Move the seeds towards the centroid of the region



# Centroidal Voronoi Diagrams

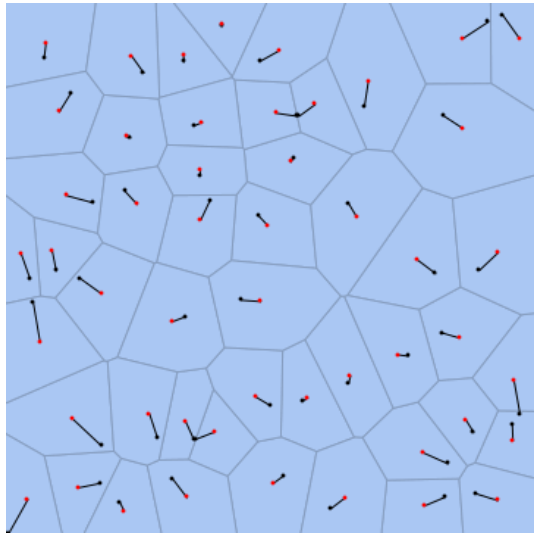
- VD generated from random seeds are not well placed around the seeds.
- Move the seeds toward the centroid of the region
- Recompute VD

# Centroidal Voronoi Diagrams

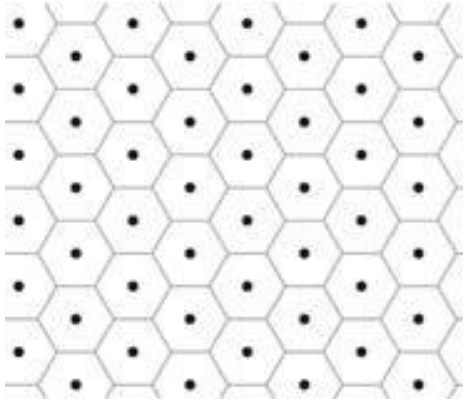
- Lloyd's Relaxation
- Until VD sites are not centroids
  - move site to centroid,
  - recalculate VD

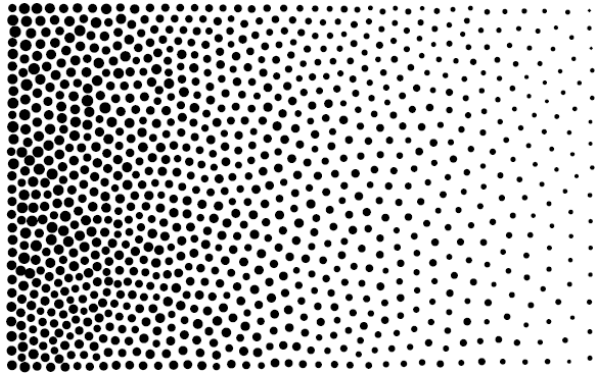
# Lloyd's method

1. generate the Voronoi tessellation $V(S)$ in $\Omega$
2. move each site $s_i \in S$ to the centroid $p_i$ of the corresponding Voronoi region $V_i \in V$;
3. if the new sites in $S$ meet some convergence criterion, then terminate; otherwise return to step 1.



Do CVT necessary lead to a good sampling?



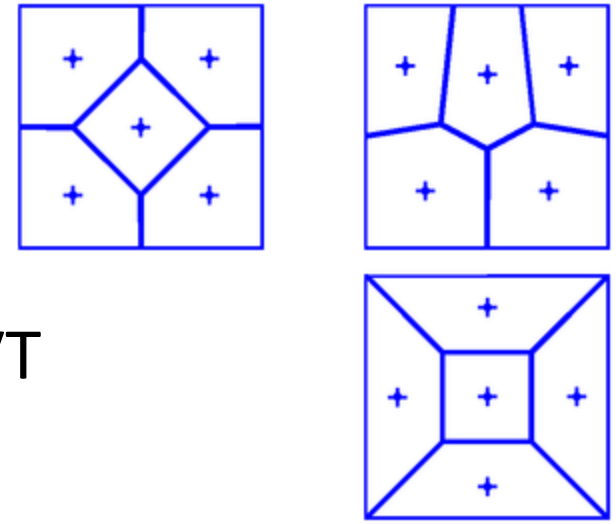


# Relaxation methods

- Methods that iteratively optimize the position of the samples w.r.t. some energy function
- **def**: Centroidal Voronoi Tessellation (Diagram).
  A CVT is a VT where each site (point) lies in the centroid of its region:

$$p_i = \frac{\int_{V_i} x\rho(x)dx}{\int_{V_i} \rho(x)dx}$$

$\rho(x)$ Some density function

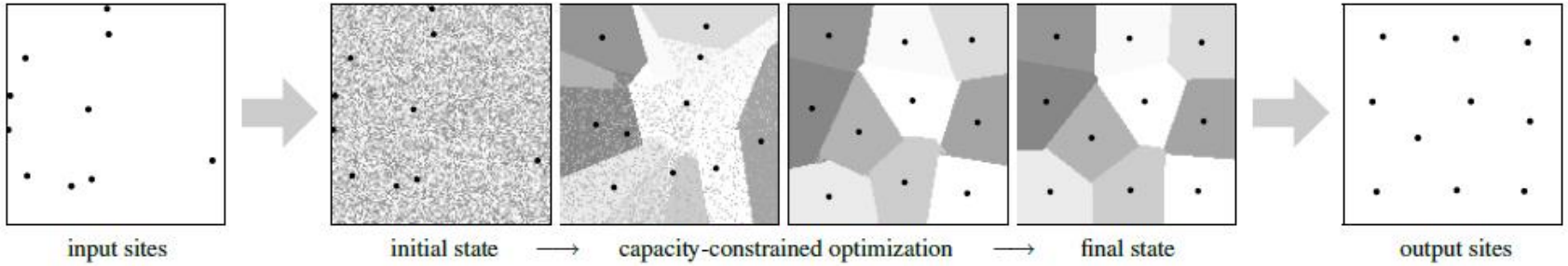


- The minimum of the energy function below is on a CVT

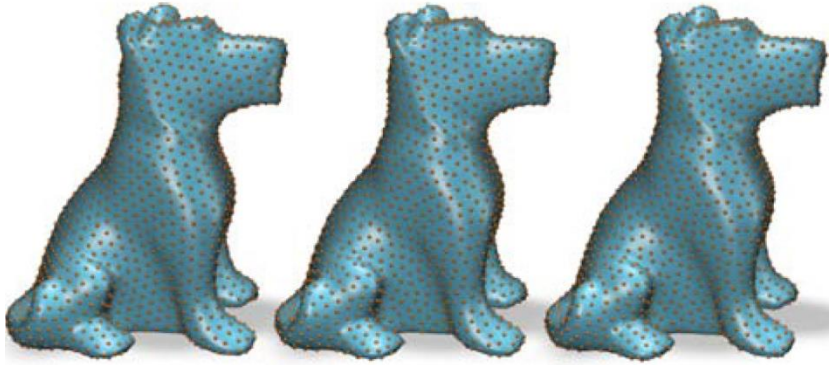
$$\mathcal{F}(S, \mathcal{V}) = \sum_{i=1}^{n} \int_{V_i} \rho(x) \left|x - s_i\right|^2 dx,$$

# CVT methods

- Methods that modify CVT to obtain better sampling properties
- Capacity Constrained CVT [balzer09] : CVT plus the constraint that each region of the VT has the same area

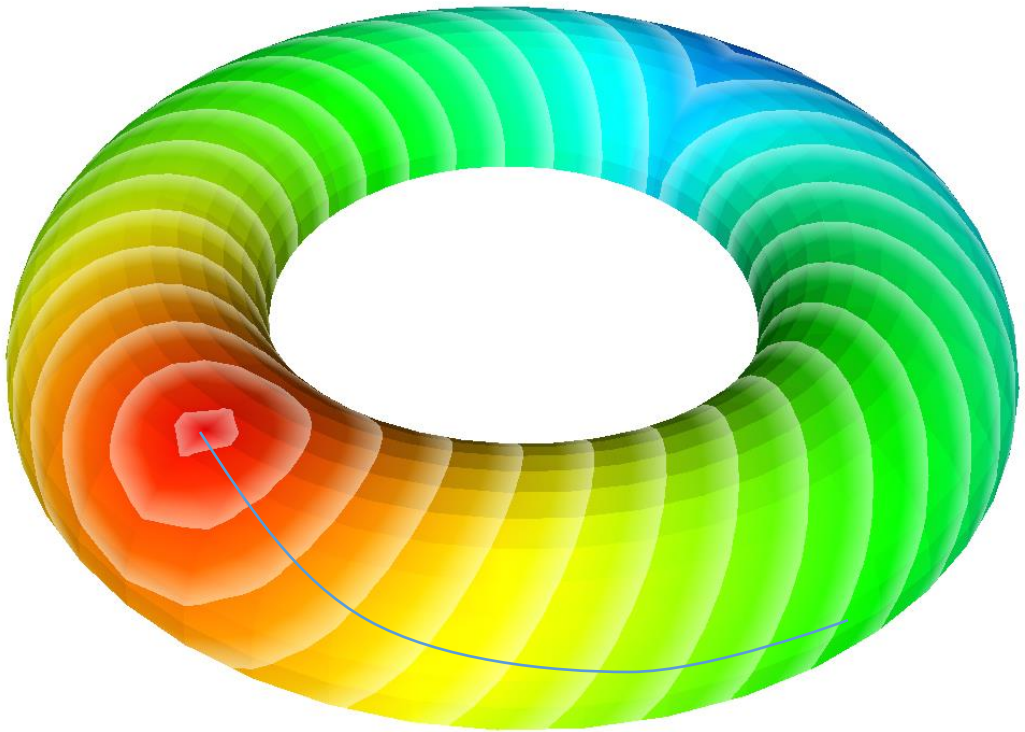



Extension to surfaces. Capacity contrained Delaunay Triangulation [chen12]

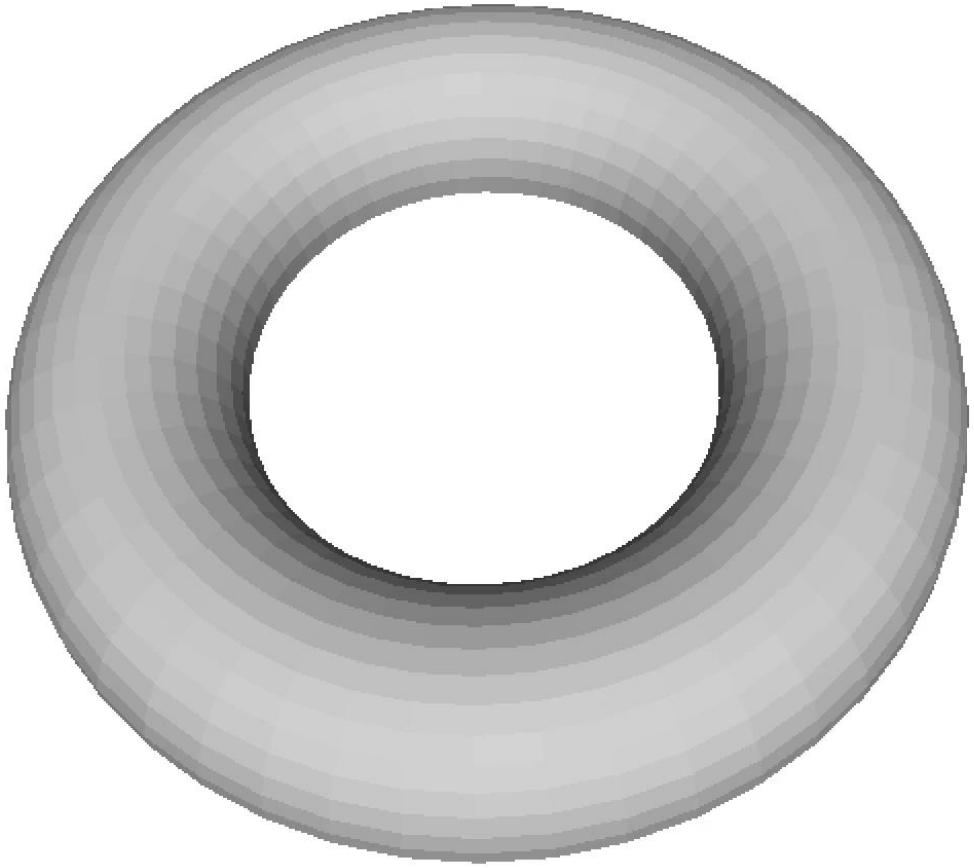


# Voronoi Diagrams over surfaces

- We just need a simple way to define the closest concept
- Geodesic:
  - Shortest path on the surface connecting two points



# Relaxing Voronoi Diagrams

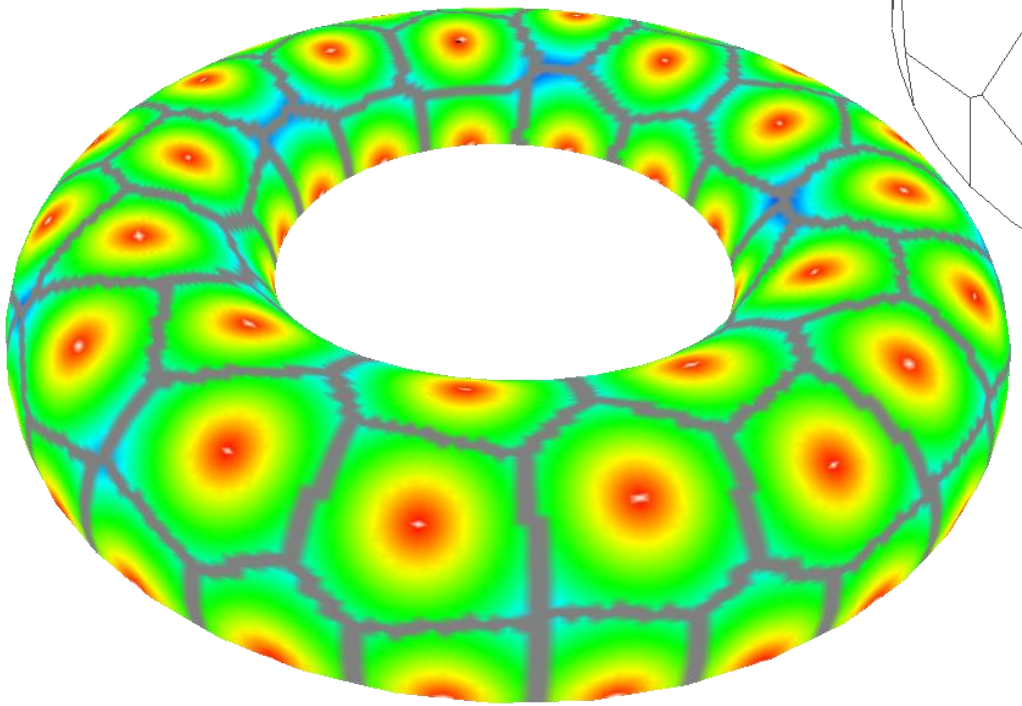


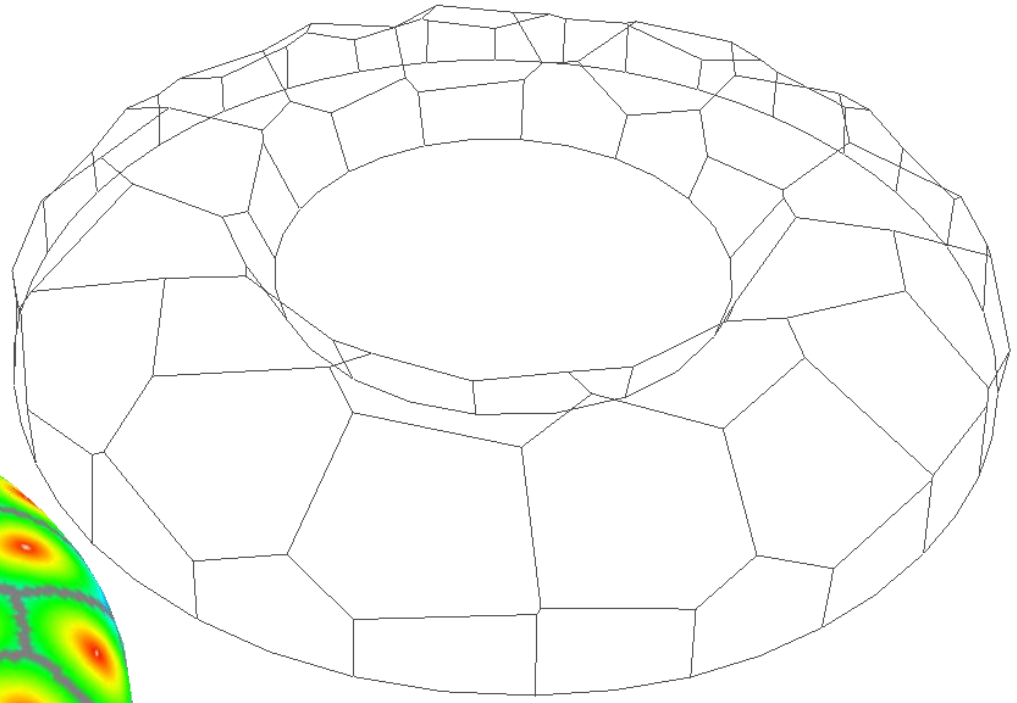
# VD: Relaxing over the surface

- Centroid is not well defined
  - it could be outside the surface
- Switch to the energy minimizing center
  - New site is the point having minimal sum of squared distances from all the points of the region.

# Voronoi Diagrams: Boundary

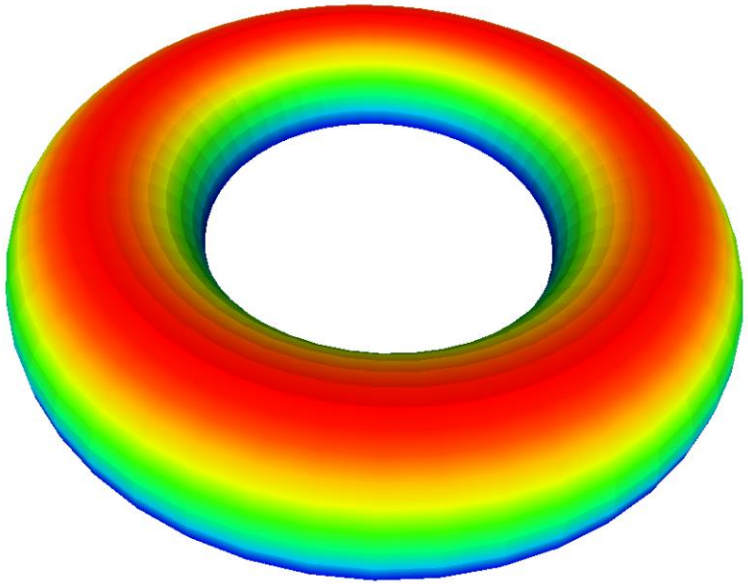
- Constraining seeds on the boundary during relaxation

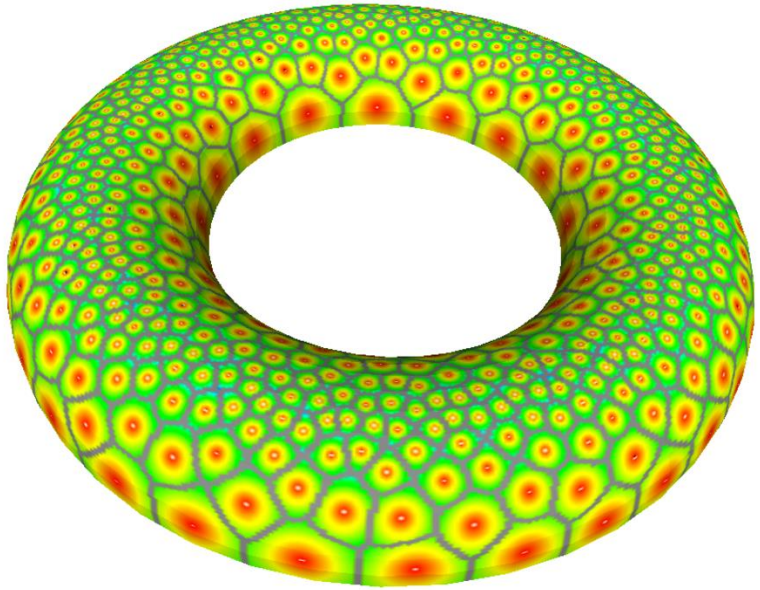




# Biasing the Voronoi Diagram

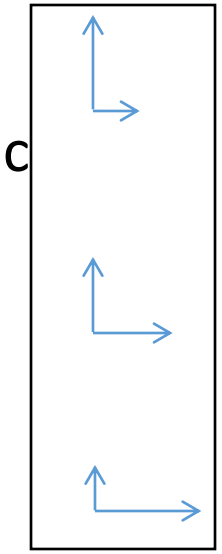
- The **density** of the pattern can be controlled
- Weight the distance according to a scalar field

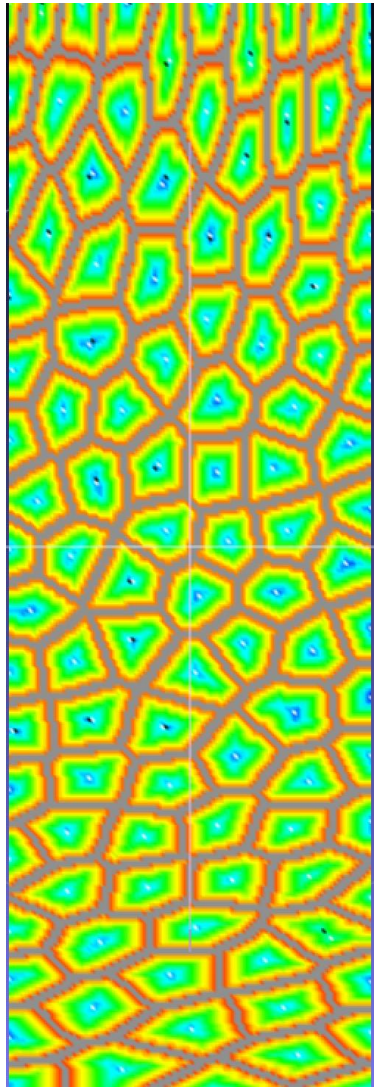




# Biasing the VD

- The **orientation** can also be controlled
- You can bias the shape of the cells: we use a frame field to define deformation
- General solution:
  - Deform the space of the domain until the defined metric is isotropic





# References


- [cook86] R. L. Cook, \Stochastic sampling in computer graphics," ACM Trans. on Graphics, vol. 5, no. 1, 1986
- [Lagae08] A. Lagae and P. Dutre, “A comparison of methods for generating Poisson disk distributions,« Computer Graphics Forum, vol. 27, no. 1,2008
- [Dunbar06] D. Dunbar and G. Humphreys, “A spatial data structure for fast Poisson-disk sample generation," ACM Trans. on Graphics (Proc. SIGGRAPH), vol. 25, no. 3, 2006
- [white07] K. B. White, D. Cline, and P. K. Egbert,”Poisson disk point sets by hierarchical dart throwing," in Proceedings of the IEEE Symposium on Interactive Ray Tracing 2007.
- [cline09] D. Cline, S. Jeschke, A. Razdan, K. White,and P. Wonka, “Dart throwing on surfaces,“ Computer Graphics Forum (Proc.EGSR), vol. 28, no. 4, 2009.
- [corsini12] M. Corsini, P. Cignoni, and R. Scopigno, Efficient and exible sampling with blue noise properties of triangular meshes," IEEE Trans. on Vis. and Comp. Graphics, vol. 18, no. 6, 2012
- [balzer09] M. Balzer, T. Schl□omer, and O. Deussen, «Capacity-constrained point distributions: A variant of Lloyd's method," ACM Trans. on Graphics (Proc. SIGGRAPH), vol. 28, no. 6, 2009
- [chen12] Z. Chen, Z. Yuan, Y.-K. Choi, L. Liu, and W. Wang, «Variational blue noise sampling,« IEEE Trans. on Vis. and Comp. Graphics, vol. 18, no. 10, pp. 1784. 2012.
- A Survey of Blue-Noise Sampling and Its Applications Dong-Ming Yan, Jian-Wei Guo, Bin Wang, Xiao-Peng Zhang & Peter Wonka Journal of Computer Science and Technology volume 30, pages439–452(2015)